

On variants of  $H$ -measures and compensated compactness <sup>☆</sup>

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**Abstract**

We introduce new variant of  $H$ -measures defined on spectra of general algebra of test symbols and derive the localization properties of such  $H$ -measures. Applications for the compensated compactness theory are given. In particular, we present new compensated compactness results for quadratic functionals in the case of general pseudo-differential constraints. The case of inhomogeneous second order differential constraints is also studied.

*Keywords:*

algebra of admissible symbols,  $H$ -measures, localization principles, compensated compactness  
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**1. Introduction**

Let

$$F(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u(x) dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transformation extended as a unitary operator on the space  $u(x) \in L^2(\mathbb{R}^n)$ , let  $S = S^{n-1} = \{ \xi \in \mathbb{R}^n \mid |\xi| = 1 \}$  be the unit sphere in  $\mathbb{R}^n$ . Denote by  $u \rightarrow \bar{u}$ ,  $u \in \mathbb{C}$  the complex conjugation.

The concept of an  $H$ -measure corresponding to some sequence of vector-valued functions bounded in  $L^2(\Omega)$ , where  $\Omega \subset \mathbb{R}^n$  is an open domain, was introduced by Tartar [9] and Ger ard [4] on the basis of the following result. For  $r \in \mathbb{N}$  let  $U_r(x) = (U_r^1(x), \dots, U_r^N(x)) \in L^2(\Omega, \mathbb{R}^N)$  be a sequence weakly convergent to the zero vector.

**Proposition 1.1** (see Theorem 1.1 in [9]). *There exists a family of complex Borel measures  $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$  in  $\Omega \times S$  and a subsequence of  $U_r(x)$  (still denoted  $U_r$ ) such that*

$$\langle \mu^{\alpha\beta}, \Phi_1(x) \overline{\Phi_2(x)} \psi(\xi) \rangle = \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(U_r^\alpha \Phi_1)(\xi) \overline{F(U_r^\beta \Phi_2)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi \quad (1.1)$$

for all  $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$  and  $\psi(\xi) \in C(S)$ .

Here and in the sequel we use notations  $C_0(\Omega)$  for the space of continuous functions on  $\Omega$  with compact supports.

The family  $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$  is called the  $H$ -measure corresponding to  $U_r(x)$ .

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In [1] the new concept of parabolic  $H$ -measures was suggested. This concept was extended in [6], where the notion of ultra-parabolic  $H$ -measures was introduced. Suppose that  $X \subset \mathbb{R}^n$  is a linear subspace,  $X^\perp$  is its orthogonal complement,  $P_1, P_2$  are orthogonal projections on  $X, X^\perp$ , respectively. We denote for  $\xi \in \mathbb{R}^n$   $\tilde{\xi} = P_1\xi$ ,  $\bar{\xi} = P_2\xi$ , so that  $\tilde{\xi} \in X$ ,  $\bar{\xi} \in X^\perp$ ,  $\xi = \tilde{\xi} + \bar{\xi}$ . Let  $S_X = \{ \xi \in \mathbb{R}^n \mid |\tilde{\xi}|^2 + |\bar{\xi}|^4 = 1 \}$ . Then  $S_X$  is a compact smooth manifold of codimension 1; in the case when  $X = \{0\}$  or  $X = \mathbb{R}^n$ , it coincides with the unit sphere  $S = \{ \xi \in \mathbb{R}^n \mid |\xi| = 1 \}$ . Let us define a projection  $\pi_X : \mathbb{R}^n \setminus \{0\} \rightarrow S_X$  by

$$\pi_X(\xi) = \frac{\tilde{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} + \frac{\bar{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}}.$$

Remark that in the case when  $X = \{0\}$  or  $X = \mathbb{R}^n$ ,  $\pi_X(\xi) = \xi/|\xi|$  is the orthogonal projection on the sphere. With the notations from Proposition 1.1, the following extension holds:

**Proposition 1.2** (see [6, 7]). *There exists a family of complex Borel measures  $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$  in  $\Omega \times S_X$  and a subsequence  $U_r(x) = U_k(x)$ ,  $k = k_r$ , such that*

$$\langle \mu^{\alpha\beta}, \Phi_1(x) \overline{\Phi_2(x)} \psi(\xi) \rangle = \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(U_r^\alpha \Phi_1)(\xi) \overline{F(U_r^\beta \Phi_2)(\xi)} \psi(\pi_X(\xi)) d\xi \quad (1.2)$$

for all  $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$  and  $\psi(\xi) \in C(S_X)$ .

The family  $\mu = \{\mu^{\alpha\beta}\}_{\alpha,\beta=1}^N$  we shall call an ultra-parabolic  $H$ -measure corresponding to  $U_r(x)$ .

In paper [7] the localization properties of ultra-parabolic  $H$ -measures were applied to extend the compensated compactness theory [5, 8] for weakly convergent sequences  $u_r \in L_{loc}^p(\Omega, \mathbb{R}^N)$  to the case when the differential constraints may contain second-order terms while all the coefficients are variable. We describe the results of [7] in the particular case  $p = 2$ . Thus, assume that a sequence  $u_r \in L_{loc}^2(\Omega, \mathbb{R}^N)$  converges weakly to a vector-function  $u(x)$  as  $r \rightarrow \infty$  and satisfies the condition that the sequences

$$\sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k} (a_{s\alpha k} u_{\alpha r}) + \sum_{\alpha=1}^N \sum_{k,l=\nu+1}^n \partial_{x_k x_l} (b_{s\alpha kl} u_{\alpha r}), \quad s = 1, \dots, m \quad (1.3)$$

are pre-compact in the anisotropic Sobolev space  $W_{2,loc}^{-1,-2}(\Omega)$  (the parameter  $-1$  corresponds to the first  $\nu$  variables  $x_1, \dots, x_\nu$  while the parameter  $-2$  corresponds to the remaining variables  $x_{\nu+1}, \dots, x_n$ ). Here  $\nu$  is an integer number between 0 and  $n$ , and the coefficients  $a_{s\alpha k} = a_{s\alpha k}(x)$ ,  $b_{s\alpha kl} = b_{s\alpha kl}(x)$  are assumed to be continuous on  $\Omega$ .

We introduce the set  $\Lambda$  (here  $i = \sqrt{-1}$ ):

$$\Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \xi \in \mathbb{R}^n, \xi \neq 0 : \right. \\ \left. \sum_{\alpha=1}^N \left( i \sum_{k=1}^{\nu} a_{s\alpha k}(x) \xi_k - \sum_{k,l=\nu+1}^n b_{s\alpha kl}(x) \xi_k \xi_l \right) \lambda_\alpha = 0 \quad \forall s = 1, \dots, m \right\}. \quad (1.4)$$

Consider the quadratic form  $q(x, u) = Q(x)u \cdot u$ , where  $Q(x)$  is a symmetric matrix with coefficients  $q_{\alpha\beta}(x) \in C(\Omega)$ ,  $\alpha, \beta = 1, \dots, N$  and  $u \cdot v$  denotes the scalar multiplication on  $\mathbb{R}^N$ .

The form  $q(x, u)$  can be extended as Hermitian form on  $\mathbb{C}^N$  by the standard relation

$$q(x, u) = \sum_{\alpha, \beta=1}^N q_{\alpha\beta}(x) u_{\alpha} \overline{u_{\beta}}.$$

Now, let the sequence  $q(x, u_r) \rightharpoonup v$  as  $r \rightarrow \infty$  weakly in  $\mathcal{D}'(\Omega)$ . Since this sequence is bounded in  $L^1_{loc}(\Omega)$  then, passing to a subsequence if necessary, we may claim that  $v$  is a locally finite measure on  $\Omega$  ( i.e.,  $v \in M_{loc}(\Omega)$  ), and  $q(x, u_r) \rightharpoonup v$  weakly in  $M_{loc}(\Omega)$ . The following result was established in [7].

**Theorem 1.1.** *Assume that  $q(x, \lambda) \geq 0$  for all  $\lambda \in \Lambda(x)$ ,  $x \in \Omega$ . Then  $q(x, u(x)) \leq v$  ( in the sense of measures ).*

In the case  $\nu = n$  when the second order terms in (1.3) are absent and all the coefficients are constant the statement of Theorem 1.1 is the classical Tartar-Murat compensated compactness.

In this paper we generalize the result of Theorem 1.1 to the case when the degeneration subspaces  $X_s$  in constraints (1.3) may depend on  $s$  and give some applications.

For that, we introduce the general variant of  $H$ -measures by extension of a class of admissible test functions  $\psi(\xi)$ . We will describe this class in the next section.

## 2. Algebra of admissible symbols

Let us denote by  $B_{\Phi}$  and  $A_{\psi}$  the bounded pseudodifferential operators on  $L^2(\mathbb{R}^n)$  with symbols  $\Phi(x), \psi(\xi) \in L^{\infty}(\mathbb{R}^n)$ , respectively, that is,

$$B_{\Phi}u(x) = \Phi(x)u(x), \quad F(A_{\psi}u)(\xi) = \psi(\xi)F(u)(\xi).$$

We introduce the subalgebra  $A$  of the algebra  $L^{\infty}(\mathbb{R}^n)$ , consisting of bounded measurable functions  $\psi(\xi)$  on  $\mathbb{R}^n$  such that the commutators  $[A_{\psi}, B_{\Phi}]$  are compact operators in  $L^2(\mathbb{R}^n)$  for all  $\Phi(x) \in C_0(\mathbb{R}^n)$ . Let  $A_0 = L^{\infty}_0$  be a subspace of  $L^{\infty}(\mathbb{R}^n)$  consisting of functions  $\psi(\xi)$  vanishing at infinity:  $\text{ess lim}_{|\xi| \rightarrow \infty} \psi(\xi) = 0$ .

**Lemma 2.1.** *For every  $\Phi(x) \in C_0(\mathbb{R}^n)$ ,  $\psi(\xi) \in A_0$  the operators  $A_{\psi}B_{\Phi}$ ,  $B_{\Phi}A_{\psi}$  are compact in  $L^2(\mathbb{R}^n)$ .*

PROOF. First, assume that  $\psi(\xi) \in L^{\infty}(\mathbb{R}^n)$  is a function with compact support  $K = \text{supp } \psi \subset \mathbb{R}^n$ . Let  $u_k$ ,  $k \in \mathbb{N}$ , be a sequence in  $L^2(\mathbb{R}^n)$ , weakly convergent to zero:  $u_k \xrightarrow{k \rightarrow \infty} 0$ . We have to prove that  $A_{\psi}B_{\Phi}u_k \xrightarrow{k \rightarrow \infty} 0$  in  $L^2(\mathbb{R}^n)$  (strongly). Since  $B_{\Phi}u_k = \Phi(x)u_k(x) \xrightarrow{k \rightarrow \infty} 0$  weakly in  $L^1(\mathbb{R}^n)$ , then

$$F(B_{\Phi}u_k)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u_k(x) \Phi(x) dx \xrightarrow{k \rightarrow \infty} 0$$

for all  $\xi \in \mathbb{R}^n$ , and

$$|F(B_{\Phi}u_k)(\xi)| \leq \|\Phi u_k\|_1 \leq C = \|\Phi\|_2 \sup_{k \in \mathbb{N}} \|u_k\|_2 < \infty.$$

Then, by the Lebesgue dominated convergence theorem, we claim that

$$\|A_{\psi}B_{\Phi}u_k\|_2^2 = \int_K |F(B_{\Phi}u_k)(\xi) \psi(\xi)|^2 d\xi \rightarrow 0$$

as  $k \rightarrow \infty$ , that is,  $A_\psi B_\Phi u_k \rightarrow 0$  in  $L^2(\mathbb{R}^n)$ . We see that the operator  $A_\psi B_\Phi$  transforms weakly convergent sequences in  $L^2$  to strongly convergent ones. Hence, this operator is compact.

In the general case  $\psi(\xi) \in A_0$  we introduce the sequence  $\psi_m(\xi) = \psi(\xi)\theta(m - |\xi|)$ ,  $m \in \mathbb{N}$ , where  $\theta(r) = \begin{cases} 0, & r \leq 0, \\ 1, & r > 0 \end{cases}$  is the Heaviside function. Then

$$\|\psi_m - \psi\|_\infty = \operatorname{ess\,sup}_{|\xi| \geq m} |\psi(\xi)| \rightarrow 0$$

as  $m \rightarrow \infty$ , and therefore the operator norms

$$\|A_{\psi_m} - A_\psi\| = \|\psi_m - \psi\|_\infty \xrightarrow{m \rightarrow \infty} 0.$$

This implies that  $A_{\psi_m} B_\Phi \rightarrow A_\psi B_\Phi$  as  $m \rightarrow \infty$  in the algebra of bounded linear operators on  $L^2(\mathbb{R}^n)$ . The functions  $\psi_m(\xi)$  have compact supports and it has been already proven that the operators  $A_{\psi_m} B_\Phi$  are compact. We conclude that  $A_\psi B_\Phi$  is a compact operator, as the limit of the sequence of compact operators  $A_{\psi_m} B_\Phi$ .

In order to prove compactness of  $B_\Phi A_\psi$ , observe that this operator is conjugate to  $A_{\bar{\psi}} B_{\bar{\Phi}} = (A_\psi)^*(B_\Phi)^*$ . As we have already established, the operator  $A_{\bar{\psi}} B_{\bar{\Phi}}$  is compact. Therefore, the operator  $B_\Phi A_\psi = (A_{\bar{\psi}} B_{\bar{\Phi}})^*$  is compact as well. The proof is complete.

In view of Lemma 2.1 we find that for  $\psi(\xi) \in A_0$  the commutator  $[A_\psi, B_\Phi] = A_\psi B_\Phi - B_\Phi A_\psi$  is a compact operator in  $L^2(\mathbb{R}^n)$  for all  $\Phi(x) \in C_0(\mathbb{R}^n)$ . In particular  $A_0 \subset A$ . It is clear that  $A_0$  is a closed ideal in  $A$ . We denote by  $\mathcal{A} = A/A_0$  the correspondent quotient algebra. Clearly,  $\mathcal{A}$  is a commutative Banach  $C^*$ -algebra (subject to the involution defined by complex conjugation) equipped with the factor-norm (we identify the class  $[\psi] \in \mathcal{A}$  with the corresponding representative function  $\psi(\xi)$ )

$$\|\psi\| = \operatorname{ess\,limsup}_{\xi \rightarrow \infty} |\psi(\xi)| = \lim_{R \rightarrow \infty} \operatorname{ess\,sup}_{|\xi| > R} |\psi(\xi)|.$$

Therefore, the Gelfand transform  $\psi(\xi) \rightarrow \hat{\psi}(\eta)$  is an isomorphism of  $\mathcal{A}$  into the algebra  $C(\mathcal{S})$  of continuous functions on the spectrum  $\mathcal{S}$  of  $\mathcal{A}$ .

We introduce the order in  $\mathcal{A}$  generated by the cone of nonnegative functions, that is, a class  $a \geq 0$  if and only if there exists a real nonnegative function  $\psi \in a$ , i.e.,  $a = [\psi]$ . As is easy to verify, for  $a, b \in \mathcal{A}$ ,  $a, b \geq 0$ , and  $\alpha, \beta \in [0, +\infty)$   $\alpha a + \beta b \geq 0$   $ab \geq 0$ . As usual, we say that  $a_1 \geq a_2$  if  $a_1 - a_2 \geq 0$ . It turns out that the Gelfand transform is monotone, that is, the following statement is fulfilled.

**Lemma 2.2.** *The class  $a = [\psi] \geq 0$  if and only if  $\hat{\psi}(\eta) \geq 0$  for all  $\eta \in \mathcal{S}$ .*

PROOF. If  $\hat{\psi}(\eta) \geq 0$  for all  $\eta \in \mathcal{S}$  then the function  $\alpha(\eta) = (\hat{\psi}(\eta))^{1/2}$  is well-defined and continuous on  $\mathcal{S}$ . Therefore, there exists a unique class  $b = [\beta(\xi)] \in \mathcal{A}$  such that  $\alpha(\eta) = \hat{\beta}(\eta)$ . Since the Gelfand transform satisfies the property  $\widehat{\widehat{\psi}}(\eta) = \hat{\psi}(\eta)$ , we see that  $\widehat{b\bar{b}}(\eta) = (\alpha(\eta))^2 = \hat{\psi}(\eta)$  and the equality  $a = [\psi] = b\bar{b} = [|\beta|^2]$  follows. This equality implies that  $a \geq 0$ .

Conversely, let  $a = [\psi] \geq 0$ . Since  $a = \bar{a}$ , the function  $\hat{\psi}(\eta)$  is real. We define the real nonnegative functions  $\hat{\psi}^\pm(\eta) = \max(0, \pm \hat{\psi}(\eta)) \in C(\mathcal{S})$ . Then, there exist classes  $a^\pm = [\psi^\pm]$  such that  $\widehat{\psi^\pm}(\eta) = \hat{\psi}^\pm(\eta)$ . As we have already established,  $a^\pm \geq 0$ . Since  $\hat{\psi}(\eta) = \hat{\psi}^+(\eta) - \hat{\psi}^-(\eta)$ , and  $\hat{\psi}^+(\eta) \cdot \hat{\psi}^-(\eta) = 0$ , the same is true for the  $a^\pm$ :  $a = a^+ - a^-$ ,  $a^+ a^- = 0$ . Therefore,

$-aa^- = (a^-)^2 \geq 0$ . On the other hand,  $aa^- \geq 0$ , as a product of nonnegative elements. We conclude that  $(a^-)^2 = -aa^- = 0$  and, therefore,  $a^- = 0$ . But this means that  $\hat{\psi}^-(\eta) = 0$  and implies nonnegativity of  $\hat{\psi}(\eta)$ :  $\hat{\psi}(\eta) = \hat{\psi}^+(\eta) \geq 0$ . This completes the proof.

As follows from [6, Lemma 2], functions  $\psi(\pi_X(\xi))$  belong to the algebra  $A$  for each  $\psi \in C(S_X)$ . Hence, the algebra of quasi-homogeneous functions

$$A_X = \{ \psi(\pi_X(\xi)) \mid \psi \in C(S_X) \}$$

is a closed  $C^*$ -subalgebra of  $\mathcal{A}$  and its spectrum coincides with  $S_X$ . The embedding  $A_X \subset \mathcal{A}$  yields the continuous projection of the spectra  $p_X : \mathcal{S} \rightarrow S_X$ . One of our aims is to formulate localization properties for  $H$ -measures corresponding to sequences satisfying general second order differential constraints. For this, we need to find simple necessary and sufficient conditions for a family of vectors  $\{\xi_X\}_{X \subset \mathbb{R}^n}$  to satisfy the property  $\xi_X = p_X(\eta)$  for all  $X \subset \mathbb{R}^n$ , where  $\eta \in \mathcal{S}$ . The following statement holds.

**Proposition 2.1.** *Assume that  $\eta \in \mathcal{S}$  and for  $X \subset \mathbb{R}^n$  let  $p_X(\eta) = (\tilde{\xi}_X, \bar{\xi}_X) \in X \oplus X^\perp$ . Then there exist a unique orthonormal system  $\{\zeta_1, \dots, \zeta_m\}$  in  $\mathbb{R}^n$  and an integer  $d \in \{m-1, m\}$  such that*

- (i)  $\tilde{\xi}_X \neq 0 \Leftrightarrow X \supset \tilde{X} \doteq \mathcal{L}(\zeta_1, \dots, \zeta_d)$  (this is a linear span of vectors  $\zeta_1, \dots, \zeta_d$ ). Besides, if  $\tilde{\xi}_X \neq 0$ , then  $\tilde{\xi}_X \uparrow\uparrow \zeta_1$ ;
- (ii)  $\bar{\xi}_X \neq 0 \Leftrightarrow X \not\supset \bar{X} \doteq \mathcal{L}(\zeta_1, \dots, \zeta_m)$ . Besides, if  $\bar{\xi}_X \neq 0$ , then  $\bar{\xi}_X \uparrow\uparrow \text{pr}_{X^\perp} \zeta_{k(X)}$ , where  $k(X) = \min\{k = 1, \dots, m \mid \zeta_k \notin X\}$ .

PROOF. We divide the proof into 6 steps.

**1st Step.**

We introduce the set  $\tilde{\mathcal{L}}$  of all subspaces  $X \subset \mathbb{R}^n$  such that  $\tilde{\xi}_X \neq 0$ . Let us show that  $\tilde{\mathcal{L}}$  contains the smallest space. For that, we first prove that the intersection  $X_1 \cap X_2$  of two spaces  $X_1, X_2 \in \tilde{\mathcal{L}}$  lays in  $\tilde{\mathcal{L}}$  as well. We denote  $X_0 = X_1 \cap X_2$ ,  $X_{10} = X_1 \ominus X_0 = \{x \in X_1 : x \perp X_0\}$ ,  $X_{20} = X_2 \ominus X_0$ . Then we have the following representations

$$\mathbb{R}^n = X_0 \oplus X_{10} \oplus X_1^\perp = X_0 \oplus X_{20} \oplus X_2^\perp. \quad (2.1)$$

Let

$$\xi = \xi_0 + \xi_1 + \xi_3 = \xi_0 + \xi_2 + \xi_4 \quad (2.2)$$

be orthogonal decompositions of a vector  $\xi \in \mathbb{R}^n$  corresponding to (2.1). Here  $\xi_0 \in X_0$ ,  $\xi_1 \in X_{10}$ ,  $\xi_2 \in X_{20}$ ,  $\xi_3 \in X_1^\perp$ , and  $\xi_4 \in X_2^\perp$ . We introduce the functions

$$f_1(\xi) = \frac{|\xi_0|^2 + |\xi_1|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4}, \quad f_1(\xi) = \frac{|\xi_0|^2 + |\xi_2|^2}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4}$$

defined on  $\mathbb{R}^n \setminus \{0\}$ . Obviously,  $f_1 \in A_{X_1} \subset \mathcal{A}$ ,  $f_2 \in A_{X_2} \subset \mathcal{A}$ , and  $\widehat{f_1}(\eta) = |\tilde{\xi}_{X_1}|^2 \neq 0$ ,  $\widehat{f_2}(\eta) = |\tilde{\xi}_{X_2}|^2 \neq 0$ . We define the subspace  $Y \subset X_1^\perp \oplus X_2^\perp$  consisting of pairs  $(\xi_3, \xi_4)$  such that  $\xi_1 + \xi_3 = \xi_2 + \xi_4$  for some vectors  $\xi_1 \in X_{10}$ ,  $\xi_2 \in X_{20}$ . Observe that the vectors  $\xi_1$ ,  $\xi_2$  are uniquely defined by the above equality. Indeed, if  $\xi'_1 + \xi_3 = \xi'_2 + \xi_4$  for some other vectors  $\xi'_1 \in X_{10}$ ,  $\xi'_2 \in X_{20}$  then  $\xi_1 - \xi'_1 = \xi_2 - \xi'_2 \in X_{10} \cap X_{20} = \{0\}$  and we conclude that  $\xi_1 = \xi'_1$ ,  $\xi_2 = \xi'_2$ . Thus, we can define the linear maps  $A_1 : Y \rightarrow X_{10}$ ,  $A_2 : Y \rightarrow X_{20}$  such that  $A_1(\xi_3, \xi_4) = \xi_1$ ,  $A_2(\xi_3, \xi_4) = \xi_2$ . Since these maps are continuous, we can find a positive constant  $C$  such that

$$|A_i(\xi_3, \xi_4)|^2 \leq C(|\xi_3|^2 + |\xi_4|^2) \quad \text{for all } (\xi_3, \xi_4) \in Y. \quad (2.3)$$

Then

$$\begin{aligned} f_1(\xi) &\leq \frac{|\xi_0|^2 + C(|\xi_3|^2 + |\xi_4|^2)}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4} \leq \frac{|\xi_0|^2 + C|\xi_4|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4} + \alpha_1(\xi), \\ f_2(\xi) &\leq \frac{|\xi_0|^2 + C(|\xi_3|^2 + |\xi_4|^2)}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4} \leq \frac{|\xi_0|^2 + C|\xi_3|^2}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4} + \alpha_2(\xi), \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} \alpha_1(\xi) &= \frac{C|\xi_3|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4} \xrightarrow{\xi \rightarrow \infty} 0, \\ \alpha_2(\xi) &= \frac{C|\xi_4|^2}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4} \xrightarrow{\xi \rightarrow \infty} 0, \end{aligned}$$

that is,  $\alpha_k(\xi) \in A_0$ ,  $j = 1, 2$ . In view of (2.4)

$$\begin{aligned} 0 &\leq f_1(\xi) \leq \frac{|\xi_0|^2 + C|\xi_4|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4}, \\ 0 &\leq f_2(\xi) \leq \frac{|\xi_0|^2 + C|\xi_3|^2}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4} \end{aligned}$$

in  $\mathcal{A}$ , which implies that in this algebra

$$0 \leq f_1(\xi)f_2(\xi) \leq \frac{(|\xi_0|^2 + C|\xi_4|^2)(|\xi_0|^2 + C|\xi_3|^2)}{(|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4)}. \quad (2.5)$$

Observe that

$$\begin{aligned} (|\xi_0|^2 + C|\xi_4|^2)(|\xi_0|^2 + C|\xi_3|^2) &\leq |\xi_0|^4 + C|\xi_4|^2(|\xi_0|^2 + C|\xi_3|^2) + C|\xi_0|^2|\xi_3|^2, \\ (|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4) &\geq (|\xi_0|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_4|^4), \end{aligned}$$

and it follows from (2.5) that

$$\begin{aligned} 0 &\leq f_1(\xi)f_2(\xi) \leq \\ &\frac{|\xi_0|^4}{(|\xi_0|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_4|^4)} + C \frac{|\xi_0|^2 + C|\xi_3|^2}{|\xi_0|^2 + |\xi_3|^4} \cdot \frac{|\xi_4|^2}{|\xi_0|^2 + |\xi_4|^4} + \\ &C \frac{|\xi_0|^2}{|\xi_0|^2 + |\xi_4|^4} \cdot \frac{|\xi_3|^2}{|\xi_0|^2 + |\xi_3|^4} = \frac{|\xi_0|^4}{(|\xi_0|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_4|^4)} + \beta(\xi), \end{aligned} \quad (2.6)$$

where  $\beta(\xi) \in A_0$ . Since

$$\begin{aligned} (|\xi_1|^2 + |\xi_3|^2)^2 &\leq (C(|\xi_3|^2 + |\xi_4|^2) + |\xi_3|^2)^2 \leq \\ (C + 1)^2(|\xi_3|^2 + |\xi_4|^2)^2 &\leq 2(C + 1)^2(|\xi_3|^4 + |\xi_4|^4), \end{aligned}$$

then

$$\begin{aligned} (|\xi_0|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_4|^4) &\geq |\xi_0|^2(|\xi_0|^2 + |\xi_3|^4 + |\xi_4|^4) \geq \\ \frac{1}{2(C + 1)^2}|\xi_0|^2 &(|\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2)^2), \end{aligned}$$

and it follows from (2.6) that in  $\mathcal{A}$

$$0 \leq f_1(\xi)f_2(\xi) \leq f_3(\xi) = \frac{2(C+1)^2|\xi_0|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2} \in A_{X_0}. \quad (2.7)$$

Taking into account monotonicity of the Gelfand transform (cf. Lemma 2.2), we derive from (2.7) that

$$0 < |\tilde{\xi}_{X_1}|^2 |\tilde{\xi}_{X_2}|^2 = \widehat{f_1}(\eta) \widehat{f_2}(\eta) \leq \widehat{f_3}(\eta) = 2(C+1)^2 |\tilde{\xi}_{X_0}|^2.$$

Hence,  $\tilde{\xi}_{X_0} \neq 0$  and  $X_0 = X_1 \cap X_2 \in \tilde{\mathcal{L}}$ . Let  $\tilde{X}$  be a subspace from  $\tilde{\mathcal{L}}$  of minimal dimension. As was already established, for each  $X \in \tilde{\mathcal{L}}$  the subspace  $X_0 = X \cap \tilde{X} \in \tilde{\mathcal{L}}$ . Since  $X_0 \subset \tilde{X}$  while  $\dim \tilde{X} \leq \dim X_0$ , we obtain that  $\tilde{X} = X_0 \subset X$ . Thus,  $X \supset \tilde{X} \forall X \in \tilde{\mathcal{L}}$ . Let us demonstrate that, conversely, any subspace  $X \supset \tilde{X}$  belongs to  $\tilde{\mathcal{L}}$  and  $\tilde{\xi}_X \uparrow \tilde{\xi}_{\tilde{X}}$ . For that, we introduce the space  $X_1 = X \ominus \tilde{X}$ , so that  $\mathbb{R}^n = \tilde{X} \oplus X_1 \oplus X^\perp$ . Denote by  $\xi_0, \xi_1, \xi_2$  the orthogonal projections of a vector  $\xi \in \mathbb{R}^n$  on the subspaces  $\tilde{X}, X_1, X^\perp$ , respectively. Then  $\xi = \xi_0 + \xi_1 + \xi_2$ . For arbitrary  $u, v \in \mathbb{R}^n$  we find

$$\begin{aligned} & \frac{u \cdot \xi_0}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} \cdot \frac{v \cdot (\xi_0 + \xi_1)}{(|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4)^{1/2}} = \\ & \frac{v \cdot \xi_0}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} \cdot \frac{u \cdot \xi_0}{(|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4)^{1/2}} + \gamma(\xi), \end{aligned}$$

where  $\gamma(\xi) = \frac{v \cdot \xi_1}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} \in A_0$ . Applying the Gelfand transform to the above equality, we obtain the equality

$$(u \cdot \tilde{\xi}_{\tilde{X}})(v \cdot \tilde{\xi}_X) = (u \cdot \text{pr}_{\tilde{X}} \tilde{\xi}_X)(v \cdot \tilde{\xi}_{\tilde{X}}). \quad (2.8)$$

Taking  $u = \tilde{\xi}_{\tilde{X}}, v \perp \tilde{X}$ , we derive from (2.8) that  $v \cdot \tilde{\xi}_X = 0$  for all  $v \perp \tilde{X}$ , which implies the inclusion  $\tilde{\xi}_X \in \tilde{X}$ . In particular,  $\text{pr}_{\tilde{X}} \tilde{\xi}_X = \tilde{\xi}_X$  and it follows from (2.8) that

$$(u \cdot \tilde{\xi}_{\tilde{X}})(v \cdot \tilde{\xi}_X) = (u \cdot \tilde{\xi}_X)(v \cdot \tilde{\xi}_{\tilde{X}}) \quad \forall u, v \in \mathbb{R}^n.$$

In view of this relation we find that  $\tilde{\xi}_X = c\tilde{\xi}_{\tilde{X}}$  for some real constant  $c$ . Further,

$$\frac{|\xi_0|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2} = \frac{|\xi_0|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4} \cdot \frac{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2}. \quad (2.9)$$

Observe that

$$\frac{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2} = g(\xi) = \frac{|\xi_0|^2 + |\xi_2|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2}$$

up to a term vanishing at infinity, and  $g(\xi) \in A_{\tilde{X}}$ . Hence, applying the Gelfand transform to (2.9), we obtain

$$0 < |\tilde{\xi}_{\tilde{X}}|^2 = |\tilde{\xi}_X|^2 \hat{g}(\eta).$$

It follows from this relation that  $\tilde{\xi}_X \neq 0$ , and the constant  $c \neq 0$ . Finally,  $c|\tilde{\xi}_{\tilde{X}}|^2 = \tilde{\xi}_X \cdot \tilde{\xi}_{\tilde{X}} = \hat{h}(\eta)$ , where

$$h(\xi) = \frac{|\xi_0|^2}{(|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4)^{1/2} (|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} \geq 0.$$

By the monotonicity of the Gelfand transform, we find that  $c > 0$ . Therefore,  $\tilde{\xi}_X \uparrow \tilde{\xi}_{\tilde{X}}$ . Denote  $\zeta_1 = \tilde{\xi}_{\tilde{X}}/|\tilde{\xi}_{\tilde{X}}| \in \tilde{X}$  (remark that  $\zeta_1 = \tilde{\xi}_{\mathbb{R}^n}$ ). Thus,

$$\tilde{\xi}_X \neq 0 \Leftrightarrow X \supset \tilde{X} \text{ and } \tilde{\xi}_X \uparrow \zeta_1. \quad (2.10)$$

### 2nd Step.

We introduce the family  $\bar{\mathcal{L}} = \{X \subset \mathbb{R}^n \mid \bar{\xi}_X = 0\}$ . Let  $X_1, X_2 \in \bar{\mathcal{L}}$ . We show that  $X_0 = X_1 \cap X_2 \in \bar{\mathcal{L}}$ . For that, we denote  $X_{10} = X_1 \ominus X_0$ ,  $X_{20} = X_2 \ominus X_0$ . Then representations (2.1) and (2.2) hold. We introduce the functions

$$g_1(\xi) = \frac{|\xi_3|^4}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4}, \quad g_2(\xi) = \frac{|\xi_4|^4}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4},$$

and remark that  $\widehat{g}_1(\eta) = \widehat{g}_2(\eta) = 0$ , in view of the condition  $\bar{\xi}_{X_1} = \bar{\xi}_{X_2} = 0$ . Since

$$\begin{aligned} h_1(\xi) &= \frac{|\xi_3|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2)^2} = g_1(\xi) \frac{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2)^2}, \\ h_2(\xi) &= \frac{|\xi_4|^4}{|\xi_0|^2 + (|\xi_2|^2 + |\xi_4|^2)^2} = g_2(\xi) \frac{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^4}{|\xi_0|^2 + (|\xi_2|^2 + |\xi_4|^2)^2}, \end{aligned}$$

then  $0 \leq h_k(\xi) \leq 2g_k(\xi)$  for  $k = 1, 2$  and sufficiently large  $|\xi|$ . Therefore,  $0 \leq \widehat{h}_k(\eta) \leq 2\widehat{g}_k(\eta) = 0$ ,  $k = 1, 2$ , and we arrive at  $\widehat{h}_k(\eta) = 0$  for  $k = 1, 2$ . Remark also that  $|\xi_1|^2 + |\xi_3|^2 = |\xi_2|^2 + |\xi_4|^2 = |\xi - \xi_0|^2$ , and, therefore,

$$p(\xi) \doteq |\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2)^2 = |\xi_0|^2 + (|\xi_2|^2 + |\xi_4|^2)^2. \quad (2.11)$$

By estimates (2.3) we see that  $|\xi_1|^2 \leq C(|\xi_3|^2 + |\xi_4|^2)$ . This inequality together with (2.11) imply that

$$\frac{|\xi_1|^2}{(p(\xi))^{1/2}} \leq C \frac{|\xi_3|^2}{(p(\xi))^{1/2}} + C \frac{|\xi_4|^2}{(p(\xi))^{1/2}} = C(h_1(\xi))^{1/2} + C(h_2(\xi))^{1/2}.$$

Therefore,

$$h(\xi) \doteq \frac{|\xi_1|^2 + |\xi_3|^2}{(p(\xi))^{1/2}} \leq (C+1)(h_1(\xi))^{1/2} + C(h_2(\xi))^{1/2}.$$

This implies that  $|\bar{\xi}_{X_0}|^2 = \widehat{h}(\eta) \leq (C+1)(\widehat{h}_1(\eta))^{1/2} + C(\widehat{h}_2(\eta))^{1/2} = 0$ . Hence,  $\bar{\xi}_{X_0} = 0$  and  $X_0 \in \bar{\mathcal{L}}$ . This statement allows to establish existence of minimal element  $\tilde{X}$  in  $\bar{\mathcal{L}}$ , in the same way as for the family  $\tilde{\mathcal{L}}$ . Namely, let  $\tilde{X}$  be an element in  $\bar{\mathcal{L}}$  of minimal dimension. Then for arbitrary  $X \in \bar{\mathcal{L}}$  the intersection  $X_0 = \tilde{X} \cap X \in \bar{\mathcal{L}}$ . Since  $X_0 \subset \tilde{X}$  while  $\dim \tilde{X} \leq \dim X_0$ , then  $\tilde{X} = X_0 \subset X$ . Hence  $\tilde{X}$  is the smallest subspace in  $\bar{\mathcal{L}}$ . Notice also that if a subspace  $X \supset \tilde{X}$  then  $X \in \bar{\mathcal{L}}$ . Indeed,  $\mathbb{R}^n = \tilde{X} \oplus X_1 \oplus X^\perp$ , where  $X_1 = X \ominus \tilde{X}$ . Therefore,  $\xi = \xi_0 + \xi_1 + \xi_2$ , with  $\xi_0, \xi_1, \xi_2$  being the orthogonal projection of  $\xi \in \mathbb{R}^n$  on the subspaces  $\tilde{X}, X_1, X^\perp$ , respectively. Let

$$\rho(\xi) = \frac{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2}.$$

Then

$$\hat{\rho}(\eta) = |\bar{\xi}_{\tilde{X}}|^2 + |\text{pr}_{X^\perp} \bar{\xi}_{\tilde{X}}|^4 = |\bar{\xi}_{\tilde{X}}|^2 = 1$$

because  $\bar{\xi}_{\tilde{X}} = 0$  while  $|\bar{\xi}_{\tilde{X}}|^2 + |\bar{\xi}_{\tilde{X}}|^4 = 1$ .



Let  $q(\xi) = \frac{|\xi_2|^4}{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4}$ . Since  $\rho(\xi)q(\xi) = \frac{|\xi_2|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2}$ , we find  $\hat{q}(\eta) = \hat{\rho}(\eta)\hat{q}(\eta) = |\text{pr}_{X^\perp} \bar{\xi}_X|^4 = 0$ , which implies that  $\bar{\xi}_X = 0$ . Thus,  $X \in \bar{\mathcal{L}}$ , and

$$\bar{\mathcal{L}} = \{ X \subset \mathbb{R}^n \mid X \supset \bar{X} \}. \quad (2.12)$$

Notice that  $|\bar{\xi}_X| = 1$ . Therefore,  $\bar{X} \in \bar{\mathcal{L}}$  and, in view of (2.10),  $\bar{X} \supset \tilde{X}$ .

**3rd Step.**

Assume that  $X_1 \subset X_2 \subset \mathbb{R}^n$  and  $\bar{\xi}_{X_2} \neq 0$ . We claim that  $\bar{\xi}_{X_1} \neq 0$  and  $\bar{\xi}_{X_2} \upharpoonright \zeta \doteq \text{pr}_{X_2^\perp} \bar{\xi}_{X_1}$  (that is,  $\zeta = c\bar{\xi}_{X_2}$  for some  $c \geq 0$ ).

Indeed, if  $\bar{\xi}_{X_1} = 0$  then  $X_1 \in \bar{\mathcal{L}}$ . By (2.12) we find  $X_2 \in \bar{\mathcal{L}}$ . But this contradicts to the assumption  $\bar{\xi}_{X_2} \neq 0$ . Further, let

$$p_1(\xi) = (|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2)^{1/4}, \quad p_2(\xi) = (|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^4)^{1/4},$$

where  $\xi_1 = \text{pr}_{X_1} \xi$ ,  $\xi_2 = \text{pr}_{X_2 \ominus X_1} \xi$ ,  $\xi_3 = \text{pr}_{X_2^\perp} \xi$ . Evidently, for each  $u, v \in \mathbb{R}^n$

$$\frac{u \cdot \xi_3}{p_1(\xi)} \cdot \frac{v \cdot \xi_3}{p_2(\xi)} = \frac{v \cdot \xi_3}{p_1(\xi)} \cdot \frac{u \cdot \xi_3}{p_2(\xi)}.$$

Applying the Gelfand transform to this identity, we find

$$(u \cdot \zeta)(v \cdot \bar{\xi}_{X_2}) = (v \cdot \zeta)(u \cdot \bar{\xi}_{X_2}) \quad \forall u, v \in \mathbb{R}^n, \quad (2.13)$$

where  $\zeta = \text{pr}_{X_2^\perp} \bar{\xi}_{X_1}$ . It readily follows from (2.13) that  $\zeta = c\bar{\xi}_{X_2}$  for some constant  $c \in \mathbb{R}$ . Since  $\zeta \cdot \bar{\xi}_{X_2}$  coincides with the Gelfand transform of the nonnegative symbol  $\frac{|\xi_3|^2}{p_1(\xi)p_2(\xi)}$ , we conclude that  $\zeta \cdot \bar{\xi}_{X_2} \geq 0$ , i.e.  $c \geq 0$ .

**4th Step.**

In this step we prove that for any  $X \subset \bar{X}$  the vector  $\bar{\xi}_X \in \bar{X}$ . Moreover, in the case  $X \subset \tilde{X}$ ,  $X \neq \tilde{X}$  the vector  $\bar{\xi}_X \in \tilde{X}$ .

First, we notice that if  $X = \bar{X}$ , then  $\bar{\xi}_X = 0 \in \bar{X}$ . In the remaining case  $X \neq \bar{X}$  we denote by  $\xi_1, \xi_2, \xi_3$  the orthogonal projections of  $\xi \in \mathbb{R}^n$  onto the subspaces  $X$ ,  $\bar{X} \ominus X$ ,  $\bar{X}^\perp$ , respectively, and introduce the symbols

$$a(\xi) = \frac{|\xi_3|^4}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2} \in A_X, \quad b(\xi) = \frac{|\xi_3|^4}{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^4} \in A_{\bar{X}}.$$

as is easy to verify,  $a(\xi) \leq 2b(\xi)$  for sufficiently large  $|\xi|$ , which implies

$$|\text{pr}_{\bar{X}^\perp} \bar{\xi}_X|^4 = a(\eta) \leq 2\hat{b}(\eta) = 2|\bar{\xi}_X|^4 = 0 \Rightarrow \text{pr}_{\bar{X}^\perp} \bar{\xi}_X = 0.$$

This means that  $\bar{\xi}_X \in \bar{X}$ , as was to be proved.

It remains only to consider the case when  $X \subsetneq \tilde{X}$ . Let  $\xi_1, \xi_2, \xi_3$  be the orthogonal projections of  $\xi \in \mathbb{R}^n$  onto the subspaces  $X$ ,  $\tilde{X} \ominus X$ ,  $\tilde{X}^\perp$ , respectively. We introduce the functions

$$\begin{aligned} a(\xi) &= \frac{|\xi_1|^2}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2} \in A_X, \quad b(\xi) = \frac{|\xi_1|^2}{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^4} \in A_{\tilde{X}}, \\ c(\xi) &= \frac{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^4}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2} \sim \frac{|\xi_1|^2 + |\xi_3|^4}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2}. \end{aligned}$$

Since  $X \notin \tilde{\mathcal{L}}$ , then  $\tilde{\xi}_X = 0$  and

$$\hat{c}(\eta) = |\tilde{\xi}_X|^2 + |\text{pr}_{\bar{X}^\perp} \bar{\xi}_X|^4 = |\text{pr}_{\bar{X}^\perp} \bar{\xi}_X|^4. \quad (2.14)$$

Further,  $\hat{a}(\eta) = |\tilde{\xi}_X|^2 = 0$ ,  $\hat{b}(\eta) = |\tilde{\xi}_{\bar{X}}|^2 \neq 0$ , and  $0 = \hat{a}(\eta) = \hat{b}(\eta)\hat{c}(\eta)$ . Therefore,  $\hat{c}(\eta) = 0$ , and it follows from (2.14) that  $\text{pr}_{\bar{X}^\perp} \bar{\xi}_X = 0$ , that is,  $\bar{\xi}_X \in \bar{X}$ .

**5th Step.**

Here we construct the orthonormal family  $\{\zeta_k\}_{k=1}^m$ . First, we set  $\zeta_1 = \tilde{\xi}_{\mathbb{R}^n} = \bar{\xi}_{\{0\}}$ . Assuming that the vectors  $\zeta_1, \dots, \zeta_{k-1}$  have already known, we define

$$\zeta_k = \bar{\xi}_{X_{k-1}} / |\bar{\xi}_{X_{k-1}}| \in X_{k-1}^\perp, \quad (2.15)$$

where  $X_{k-1}$  is a subspace spanned by the vectors  $\zeta_1, \dots, \zeta_{k-1}$  (notice that  $X_0 = \{0\}$ ). This definition is correct while  $X_{k-1} \subsetneq \bar{X}$  because by (2.12)  $\bar{\xi}_{X_{k-1}} \neq 0$ . As was demonstrated in the 4th step,  $\zeta_k \in \bar{X}$ . We see that the construction of  $\zeta_k$  may be continued until  $k = m = \dim \bar{X}$ . The  $m$ -dimensional subspace  $X_m \subset \bar{X}$  must coincide with  $\bar{X}$ :  $X_m = \bar{X}$ , so that  $\bar{\xi}_{X_m} = 0$ . By the construction  $\{\zeta_k\}_{k=1}^m$  is an orthonormal basis in  $\bar{X}$ . Let  $d = \dim \tilde{X}$ . Then  $1 \leq d \leq m$ . By the second statement proven in 4th Step  $\zeta_k \in \tilde{X}$  while  $X_{k-1} \subsetneq \tilde{X}$ . Since  $\zeta_1 \in \tilde{X}$  then by induction  $X_k \subset \tilde{X}$  for all  $1 \leq k \leq d$ . Comparing the dimension, we claim that  $\tilde{X} = X_d = \mathcal{L}(\zeta_1, \dots, \zeta_d)$ . As was shown in 1st step, for  $\tilde{\xi}_X \neq 0$  this vector is co-directed with  $\zeta_1$ . The proof of (i) is complete.

To complete the proof of statement (ii), we choose a subspace  $X \subset \mathbb{R}^n$  such that  $\bar{\xi}_X \neq 0$ . Then, in view of (2.12)  $X \not\supset \bar{X} = \mathcal{L}(\zeta_1, \dots, \zeta_m)$ . Therefore, there exists the vector  $\zeta_k \notin X$ . Let  $k = k(X) = \min\{k = 1, \dots, m \mid \zeta_k \notin X\}$ . Then  $X_{k-1} \subset X$ ,  $\zeta_k \notin X$ . Since  $\zeta_k \uparrow \bar{\xi}_{X_{k-1}} \neq 0$ , then by the assertion established in the 3rd Step we claim that  $\tilde{\xi}_X \uparrow \text{pr}_{X^\perp} \zeta_k \neq 0$ , as was to be proved.

Remark also that by results of the 3rd Step requirement (ii) for  $X = X_{k-1}$  implies (2.15). This readily implies that the orthonormal family  $\zeta_k$ ,  $k = 1, \dots, m$  is uniquely defined by the point  $\eta$ . The parameter  $d$  is also uniquely determined by the condition  $d = \dim \tilde{X}$ .

**6th Step.** It only remains to show that  $d \geq m - 1$ . Assuming the contrary  $d \leq m - 2$ , we see that the space  $X_1$  spanned by the vectors  $\xi_k$ ,  $k = 1, \dots, d + 1$  is a proper subspace of  $\bar{X}$ :  $\bar{X} \subsetneq X_1 \subsetneq \bar{X}$ . We extend the system  $\zeta_k$ ,  $k = 1, \dots, m$  to an orthonormal basis  $\zeta_k$ ,  $k = 1, \dots, n$  in  $\mathbb{R}^n$ . Let  $s_k = s_k(\xi)$ ,  $k = 1, \dots, n$  be coordinates of a vector  $\xi \in \mathbb{R}^n$  in this basis:  $\xi = \sum_{k=1}^n s_k \zeta_k$ .

We introduce the following functions

$$p_1(\xi) = \frac{s_1^2}{\sum_{k=1}^d s_k^2 + (\sum_{k=d+1}^n s_k^2)^2}, \quad q_1(\xi) = \frac{s_{d+2}^4}{\sum_{k=1}^{d+1} s_k^2 + (\sum_{k=d+2}^n s_k^2)^2},$$

$$p_2(\xi) = \frac{s_1^2}{\sum_{k=1}^{d+1} s_k^2 + (\sum_{k=d+2}^n s_k^2)^2}, \quad q_2(\xi) = \frac{s_{d+2}^4}{\sum_{k=1}^d s_k^2 + (\sum_{k=d+1}^n s_k^2)^2}.$$

Obviously,  $p_1, q_2 \in S_{\tilde{X}}$ ,  $p_2, q_1 \in S_{X_1}$ , and  $p_1 q_1 = p_2 q_2$ . Therefore,

$$\hat{p}_1(\eta) \hat{q}_1(\eta) = \hat{p}_2(\eta) \hat{q}_2(\eta). \quad (2.16)$$

Now observe that  $\hat{p}_1(\eta) = |\tilde{\xi}_{\bar{X}}|^2 \neq 0$ ,  $\hat{p}_2(\eta) = |\tilde{\xi}_{X_1}|^2 \neq 0$ ,  $\hat{q}_1(\eta) = |\bar{\xi}_{X_1}|^4 \neq 0$  (because  $X_1 \subsetneq \bar{X}$ ), and  $\hat{q}_2(\eta) = |\bar{\xi}_{\bar{X}} \cdot \zeta_{d+2}|^4 = 0$  because  $\bar{\xi}_{\bar{X}} \uparrow \xi_{d+1} \perp \xi_{d+2}$ . Hence  $\hat{p}_1(\eta) \hat{q}_1(\eta) \neq 0$ ,  $\hat{p}_2(\eta) \hat{q}_2(\eta) = 0$ , which contradicts (2.16). The proof is complete.

The statement of Proposition 2.1 is sharp, in the sense that for every orthonormal system  $\zeta_k$ ,  $k = 1, \dots, m$ , and an integer number  $d \in \{m-1, m\}$ , one can find a point  $\eta \in \mathcal{S}$  such that the statements (i), (ii) of Proposition 2.2 hold. To prove this assertion, we need the notion of an essential ultrafilter. We call sets  $A, B \subset \mathbb{R}^n$  equivalent:  $A \sim B$  if  $\mu(A \triangle B) = 0$ , where  $A \triangle B = (A \setminus B) \cup (B \setminus A)$  is the symmetric difference and  $\mu$  is the outer Lebesgue measure. Let  $\mathfrak{F}$  be a filter in  $\mathbb{R}^n$ . This filter is called *essential* if from the conditions  $A \in \mathfrak{F}$  and  $B \sim A$  it follows that  $B \in \mathfrak{F}$ . It is clear that an essential filter cannot include sets of null measure, since such sets are equivalent to  $\emptyset$ . Using Zorn's lemma, one can prove that any essential filter is contained in a maximal essential filter. Maximal essential filters are called essential ultrafilters.

**Lemma 2.3.** *Let  $\mathfrak{U}$  be an essential ultrafilter. Then for each  $A \subset \mathbb{R}^n$  either  $A \in \mathfrak{U}$  or  $\mathbb{R}^n \setminus A \in \mathfrak{U}$ .*

PROOF. Assuming that  $A \notin \mathfrak{U}$ , we introduce

$$\mathfrak{F} = \{ B \subset \mathbb{R}^n \mid B \cup A \in \mathfrak{U} \}.$$

Obviously,  $\mathfrak{F}$  is an essential filter,  $\mathbb{R}^n \setminus A \in \mathfrak{F}$ , and  $\mathfrak{U} \leq \mathfrak{F}$ . Since the filter  $\mathfrak{U}$  is maximal, we obtain that  $\mathfrak{U} = \mathfrak{F}$ . Hence,  $\mathbb{R}^n \setminus A \in \mathfrak{U}$ . The proof is complete.

The property indicated in Lemma 2.3 is the characteristic property of ultrafilters, see for example, [3]. Therefore, we have the following statement.

**Corollary 2.1.** *Any essential ultrafilter is an ultrafilter, i.e. a maximal element in a set of all filters.*

**Lemma 2.4.** *Let  $\mathfrak{U}$  be an essential ultrafilter, and  $f(\xi)$  be a bounded function in  $\mathbb{R}^n$ . Then there exists  $\lim_{\mathfrak{U}} f(\xi)$ . If a function  $g(\xi) = f(\xi)$  almost everywhere on  $\mathbb{R}^n$ , then there exists  $\lim_{\mathfrak{U}} g(\xi) = \lim_{\mathfrak{U}} f(\xi)$ .*

PROOF. By Corollary 2.1  $\mathfrak{U}$  is an ultrafilter. By the known properties of ultrafilters, the image  $f_*\mathfrak{U}$  is an ultrafilter on the compact  $[-M, M]$ , where  $M = \sup |f(\xi)|$ , and this ultrafilter converges to some point  $x \in [-M, M]$ . Therefore,  $\lim_{\mathfrak{U}} f(\xi) = \lim_{\mathfrak{U}} f_*\mathfrak{U} = x$ . Further, suppose that a function  $g = f$  a.e. on  $\mathbb{R}^n$ . Then the set  $E = \{\xi \in \mathbb{R}^n \mid g(\xi) \neq f(\xi)\}$  has null Lebesgue measure. Let  $V$  be a neighborhood of  $x$ . Then  $g^{-1}(V) \supset f^{-1}(V) \setminus E$ . By the convergence of the ultrafilter  $f_*\mathfrak{U}$  the set  $f^{-1}(V) \in \mathfrak{U}$ . Since  $\mathfrak{U}$  is an essential ultrafilter while  $f^{-1}(V) \setminus E \sim f^{-1}(V)$ , then  $f^{-1}(V) \setminus E \in \mathfrak{U}$ . This set is contained in  $g^{-1}(V)$ , and we claim that  $g^{-1}(V) \in \mathfrak{U}$ . Since  $V$  is an arbitrary neighborhood of  $x$ , we conclude that  $\lim_{\mathfrak{U}} g(\xi) = x$ . The proof is complete.

By the statement of Lemma 2.4, the functional  $f \rightarrow \lim_{\mathfrak{U}} f(\xi)$  is well-defined on  $L^\infty(\mathbb{R}^n)$  and it is a linear multiplicative functional on  $L^\infty(\mathbb{R}^n)$ . In other words, this functional belongs to the spectrum of algebra  $L^\infty(\mathbb{R}^n)$  (actually, this spectrum coincides with the space of such functionals).

Now we are ready to prove the sharpness of Proposition 2.1.

**Proposition 2.2.** *Let  $\zeta_k$ ,  $k = 1, \dots, m$  be an orthonormal system in  $\mathbb{R}^n$ ,  $1 \leq d \in \{m-1, m\}$ . Then there exists a point  $\eta \in \mathcal{S}$  such that the statements (i), (ii) of Proposition 2.1 hold.*

PROOF. We extend vectors  $\zeta_k$ ,  $k = 1, \dots, m$  to a basis  $\zeta_k$ ,  $k = 1, \dots, n$  in  $\mathbb{R}^n$ . Let  $\sigma_k$ ,  $k = 2, \dots, n$  be a decreasing family of positive numbers such that  $1 > \sigma_2 > \dots > \sigma_d > 1/2 \geq \sigma_{d+1} > \dots > \sigma_n > 0$ , and  $\sigma_{d+1} = 1/2$  only if  $d = m - 1$ . We introduce the sets

$$B_r = \left\{ \xi = \sum_{k=1}^n s_k \zeta_k \mid |\xi| > r, s_1 > 0, \text{ and } s_1^{\sigma_k} < s_k < 2s_1^{\sigma_k} \ \forall k = 2, \dots, n \right\},$$

$r > 0$ . It is clear that  $B_r$  are nonempty open sets in  $\mathbb{R}^n$ , which form the base of some essential filter  $\mathfrak{F}$ . Let  $\mathfrak{U}$  be an essential ultrafilter such that  $\mathfrak{F} \leq \mathfrak{U}$ . Since the limit along  $\mathfrak{U}$  is a linear multiplicative functional on  $A$  vanishing on the ideal  $A_0$ , it forms a linear multiplicative functional on  $\mathcal{A}$ , and there exists a unique element  $\eta \in \mathcal{S}$  such that  $\hat{a}(\eta) = \lim_{\mathfrak{U}} a(\xi)$  for each  $a \in \mathcal{A}$ . We will demonstrate that the element  $\eta$  satisfies conditions (i), (ii) of Proposition 2.2. Assume that a subspace  $X \not\supset \tilde{X} = \mathcal{L}(\zeta_1, \dots, \zeta_d)$ . Then there exists  $k$ ,  $1 \leq k \leq d$  such that  $\zeta_k \notin X$ . Let  $k = k(X)$  be the minimal one among such  $k$ . We denote by  $P_1, P_2$  the orthogonal projections onto the spaces  $X, X^\perp$ , respectively, and set  $v_i = P_2 \zeta_i$ . Then  $v_k \neq 0$  while  $v_i = 0$  for  $1 \leq i < k$ . If  $\xi \in B_r$ , then

$$r^2 < |\xi|^2 = \sum_{i=1}^n s_i^2 \leq s_1^2 + 2 \sum_{i=2}^n s_1^{2\sigma_i} \leq C_r s_1^2, \quad (2.17)$$

where  $C_r \rightarrow 1$  as  $r \rightarrow \infty$ . Here we take into account the condition  $\sigma_i < 1$ . In particular, it follows from (2.17) that  $s_1 > r/2$  for large  $r$ . Denote, as above,  $\tilde{\xi} = P_1 \xi$ ,  $\bar{\xi} = P_2 \xi$ . Since  $\sigma_i < \sigma_k$  for  $i > k$ ,  $s_1 > r/2 \xrightarrow{r \rightarrow \infty} \infty$ , and  $|v_k| > 0$ , we find that for sufficiently large  $r$

$$|\bar{\xi}| = \left| \sum_{i=k}^n s_i v_i \right| \geq s_k |v_k| - \sum_{i=k+1}^n s_i |v_i| \geq s_1^{\sigma_k} |v_k| - 2 \sum_{i=k+1}^n s_1^{\sigma_i} |v_i| \geq c s_1^{\sigma_k}, \quad (2.18)$$

where  $c = \text{const} > 0$ . It follows from (2.17), (2.18) that for  $\xi \in B_r$ , where  $r$  is sufficiently large

$$a(\xi) = \frac{|\tilde{\xi}|^2}{|\tilde{\xi}|^2 + |\bar{\xi}|^4} \leq \frac{|\xi|^2}{|\xi|^4} \leq \frac{C_r}{c^4} s_1^{2-4\sigma_k} \rightarrow 0 \quad (2.19)$$

as  $r \rightarrow \infty$  because  $\sigma_k > 1/2$  and  $s_1 \rightarrow \infty$  as  $r \rightarrow \infty$ . It follows from (2.19) that

$$|\tilde{\xi}_X(\eta)|^2 = \hat{a}(\eta) = \lim_{\mathfrak{U}} a(\xi) = \lim_{\mathfrak{F}} a(\xi) = 0.$$

We claim that  $\tilde{\xi}_X(\eta) = 0$ .

If  $X \supset \tilde{X}$ , then  $\bar{\xi} = \sum_{i=d+1}^n s_i v_i$ , where  $v_i = P_2 \zeta_i$ , and for  $\xi \in B_r$

$$s_1^2 \leq |\tilde{\xi}|^2 \leq |\xi|^2 = \sum_{i=1}^n s_i^2 \leq C_1 s_1^2, \quad (2.20)$$

$$|\bar{\xi}| \leq \sum_{i=d+1}^n s_i \leq C_2 s_1^{\sigma_{d+1}}, \quad C_1, C_2 = \text{const}. \quad (2.21)$$

Since  $\sigma_{d+1} \leq 1/2$ , then it follows from (2.20), (2.21) that for sufficiently large  $r$

$$a(\xi) = \frac{|\tilde{\xi}|^2}{|\tilde{\xi}|^2 + |\bar{\xi}|^4} \geq (C_1 + C_2^4)^{-1} > 0,$$

which implies that  $|\tilde{\xi}_X(\eta)|^2 = \hat{a}(\eta) = \lim_{\mathfrak{U}} a(\xi) > 0$ . Hence  $\tilde{\xi}_X = \tilde{\xi}_X(\eta) \neq 0$ . Observe also that, as follows from (2.20), (2.21),

$$\frac{s_i}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \leq cs_1^{\sigma_2-1} \xrightarrow{r \rightarrow \infty} 0, \quad i = 2, \dots, n,$$

and, therefore,

$$\tilde{\xi}_X = \lim_{\mathfrak{U}} \frac{\tilde{\xi}}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} = \left( \lim_{\mathfrak{U}} \frac{s_1}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/2}} \right) \zeta_1 \uparrow \zeta_1.$$

We conclude that condition (i) is satisfied.

To prove (ii), assume that  $X \supset \bar{X} = \mathcal{L}(\zeta_1, \dots, \zeta_m)$ . Then  $\bar{\xi} = \sum_{i=m+1}^n s_i v_i$ ,  $v_i = P_2 \zeta_i$ , which implies the estimate

$$|\bar{\xi}| \leq 2 \sum_{i=m+1}^n s_1^{\sigma_i} \leq Cs_1^{\sigma_{m+1}}, \quad C = \text{const},$$

for all  $\xi \in B_r$  with sufficiently large  $r$ . On the other hand,  $|\tilde{\xi}| \geq s_1$ . Therefore, for  $\xi \in B_r$

$$\frac{|\bar{\xi}|^4}{|\tilde{\xi}|^2 + |\bar{\xi}|^4} \leq C^4 s_1^{4\sigma_{m+1}-2} \xrightarrow{r \rightarrow \infty} 0$$

because  $\sigma_{m+1} < 1/2$ . Hence,

$$|\bar{\xi}_X|^4 = \lim_{\mathfrak{U}} \frac{|\bar{\xi}|^4}{|\tilde{\xi}|^2 + |\bar{\xi}|^4} = 0,$$

that is,  $\bar{\xi}_X = 0$ .

Now, suppose that  $X \not\supset \bar{X}$ . Then there exists  $\zeta_k \notin X$ , where  $1 \leq k \leq m$ . We chose  $k$  being the minimal one. Then  $\zeta_i \in X$ ,  $1 \leq i < k$ , and  $\bar{\xi} = \sum_{i=k}^n s_i v_i$ ,  $v_i = P_2 \zeta_i$ , which implies the estimate

$$|\bar{\xi}| \geq s_k |v_k| - \sum_{i=k+1}^n s_i |v_i| \geq s_1^{\sigma_k} |v_k| - 2 \sum_{i=k+1}^n s_1^{\sigma_i} |v_i| \geq cs_1^{\sigma_k}, \quad c = |v_k|/2 > 0, \quad (2.22)$$

for all  $\xi \in B_r$  with sufficiently large  $r$ . We use here that  $v_k \neq 0$  and  $\sigma_k > \sigma_i$  for  $i > k$ . Further,

$$|\tilde{\xi}|^2 \leq |\xi|^2 = \sum_{i=1}^n s_i^2 \leq s_1^2 + 2 \sum_{i=2}^n s_1^{2\sigma_i} \leq 2|s_1|^2 \quad (2.23)$$

for all  $\xi \in B_r$  with large  $r$ . It follows from (2.22), (2.23) and from the condition  $\sigma_k \geq \sigma_m \geq 1/2$  that

$$cs_1^{\sigma_k} \leq (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4} \leq Cs_1^{\sigma_k}, \quad C = \text{const}. \quad (2.24)$$

In view of (2.24) for all  $\xi \in B_r$  with sufficiently large  $r$

$$\frac{s_i}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}} \leq \frac{2}{c} s_1^{\sigma_i - \sigma_k} \xrightarrow{r \rightarrow \infty} 0, \quad k+1 \leq i \leq n,$$

$$\frac{1}{C} \leq \frac{s_k}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}} \leq \frac{2}{c}.$$

This implies that

$$\bar{\xi}_X(\eta) = av_k \uparrow \uparrow \text{pr}_{X^\perp} \zeta_k,$$

where

$$a = \lim_{\mathfrak{A}} \frac{s_k}{(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{1/4}} > 0,$$

and  $k = k(X) = \min\{k = 1, \dots, m \mid \zeta_k \notin X\}$ .

We see that requirement (ii) of Proposition 2.2 is also satisfied. The proof is complete.

**Remark 2.1.** For each real  $t \neq 0$  the map  $h_t(\psi)(\xi) = \psi^t \doteq \psi(t\xi)$  is an isomorphism of algebra  $\mathcal{A}$ . Indeed, it is easy to verify that

$$A_{\psi^t} = Q_t^{-1} A_\psi Q_t, \quad B_\Phi = Q_t^{-1} B_{\Phi^t} Q_t \quad \forall \psi(\xi) \in A, \Phi(x) \in C_0(\mathbb{R}^n),$$

where  $\Phi^t(x) = \Phi(tx)$ , and the operator  $Q_t$  in  $L^2(\mathbb{R}^n)$  is defined by the equality  $Q_t u(x) = u^t = u(tx)$ . Therefore, the operators  $[A_{\psi^t}, B_\Phi] = Q_t^{-1} [A_\psi, B_{\Phi^t}] Q_t$  are compact in  $L^2$  for all  $\Phi(x) \in C_0(\mathbb{R}^n)$ . This implies that  $h_t$  is well-defined on  $A$  and evidently transfers the ideal  $A_0$  into itself. This allows to define the operator  $h_t$  on the quotient algebra  $\mathcal{A} = A/A_0$ . It is clear that  $h_t$  is invertible and  $h_t^{-1} = h_{1/t}$ . Therefore, the operator  $h_t$  generates the corresponding homeomorphism of the spectrum  $\widehat{h}_t : \mathcal{S} \rightarrow \mathcal{S}$ , so that  $\widehat{\psi}(\widehat{h}_t(\eta)) = \widehat{h_t(\psi)}(\eta)$ . We denote  $\widehat{h}_t(\eta) = t\eta$ . This determines an action of the multiplicative group of  $\mathbb{R}$  on the space  $\mathcal{S}$ . If  $X$  is a subspace of  $\mathbb{R}^n$ , and  $(\tilde{\xi}(\eta), \bar{\xi}(\eta)) = p_X(\eta)$ , then it is directly verified that for each  $t \neq 0$

$$\tilde{\xi}(t\eta) = a(t, \eta) \tilde{\xi}(\eta), \quad \bar{\xi}(t\eta) = b(t, \eta) \bar{\xi}(\eta),$$

where

$$a(t, \eta) = t(t^2 |\tilde{\xi}(\eta)|^2 + t^4 |\bar{\xi}(\eta)|^4)^{-1/2}, \quad b(t, \eta) = t(t^2 |\tilde{\xi}(\eta)|^2 + t^4 |\bar{\xi}(\eta)|^4)^{-1/4}.$$

In particular,  $(b(t, \eta))^2 = ta(t, \eta)$ .

### 3. $H$ -measures and the localization property

Now, let  $\Omega \subset \mathbb{R}^n$  be an open domain and  $U_r(x) \in L_{loc}^2(\Omega, \mathbb{C}^N)$  be a sequence of generally complex-valued vector functions weakly convergent to the zero vector. Denote by  $\text{Blim}$  a generalized Banach limit (see [2]), that is, a linear functional on the Banach space  $l_\infty$  of bounded sequences such that for each real sequence  $x = \{x_r\}_{r=1}^\infty \in l_\infty$

$$\lim_{r \rightarrow \infty} x_r \leq \text{Blim}_{r \rightarrow \infty} x_r \leq \overline{\lim_{r \rightarrow \infty} x_r}$$

( we use the customary notation  $\text{Blim}_{r \rightarrow \infty} x_r$  for the the Banach limit of the sequence  $x$  ).

In order to justify the notion of  $H$ -measures, we will need the following result on representation of bilinear functionals.

**Lemma 3.1.** *Let  $X, Y$  be locally compact Hausdorff spaces, and  $F(f, g)$  be a bilinear functional on  $C_0(X) \times C_0(Y)$  such that for every compact subsets  $K_1 \subset X$ ,  $K_2 \subset Y$*

$$|F(f, g)| \leq C(K_1, K_2) \|f\|_\infty \|g\|_\infty \quad \forall f \in C_0(K_1), g \in C_0(K_2), \quad (\text{continuity}), \quad (3.1)$$

where the constant  $C(K_1, K_2)$  depends only on compacts  $K_1, K_2$ , and

$$F(f, g) \geq 0 \quad \forall f, g \geq 0 \quad (\text{nonnegativity}). \quad (3.2)$$

Then there exists a unique locally finite nonnegative Radon measure  $\mu = \mu(x, y)$  on  $X \times Y$  such that

$$F(f, g) = \int_{X \times Y} f(x)g(y) d\mu(x, y). \quad (3.3)$$

PROOF. First, we consider the case when  $X, Y$  are compact sets of Euclidean spaces:  $X \subset \mathbb{R}^m$ ,  $Y \subset \mathbb{R}^l$ . In this case the statement of Lemma 3.1 was established in [9, Lemma 1.10]. For completeness we reproduce below the proof. Assuming that  $m \geq l$ , we may suppose that  $X, Y$  are compact subsets of the same Euclidean space:  $X, Y \subset \mathbb{R}^m$ . We choose a function  $K(z) \in C_0(\mathbb{R}^m)$  such that  $K(z) \geq 0$ ,  $\text{supp } K \subset B_1 \doteq \{z \in \mathbb{R}^m \mid |z| \leq 1\}$ ,  $\int K(z) dz = 1$ , and set  $K_r(z) = r^m K(rz)$ , where  $r \in \mathbb{N}$ . Obviously, the sequence  $K_r(z)$  converges as  $r \rightarrow \infty$  to the Dirac  $\delta$ -measure  $\delta(z)$  weakly in  $\mathcal{D}'(\mathbb{R}^m)$ . For  $f(x) \in C(\mathbb{R}^m)$  we introduce the averaged functions  $f_r(p) = f * K_r(p) = \int_{\mathbb{R}^m} f(x) K_r(p - x) dx$ . By the known properties of averaged functions,  $f_r \rightarrow f$  as  $r \rightarrow \infty$  uniformly on any compact. This together with the continuity assumption implies that

$$F(f, g) = \lim_{r \rightarrow \infty} F_r(f, g), \quad (3.4)$$

where  $F_r(f, g) = F(f_r, g_r)$ , and the averaged functions

$$f_r(p) = \int_{\mathbb{R}^m} f(x) K_r(p - x) dx, \quad g_r(q) = \int_{\mathbb{R}^m} g(y) K_r(q - y) dy$$

are reduced to the sets  $X$  and  $Y$ , respectively. As it follows from the continuity of  $F$ ,

$$F_r(f, g) = F(f_r, g_r) = \int_{\mathbb{R}^m \times \mathbb{R}^m} f(x)g(y) \alpha_r(x, y) dx dy, \quad (3.5)$$

where

$$\alpha_r(x, y) = F(K_r(p - x), K_r(q - y)).$$

It is easy to verify that  $\alpha_r(x, y) \in C_0(\mathbb{R}^m \times \mathbb{R}^m)$ ,  $\text{supp } \alpha_r \subset X_r \times Y_r$ , where  $X_r = X + B_{1/r}$ ,  $Y_r = Y + B_{1/r}$ ,  $r \in \mathbb{N}$ , and by  $B_\rho$  we denote the closed ball of radius  $\rho$  centered at zero:  $B_\rho = \{z \in \mathbb{R}^m \mid |z| \leq \rho\}$ . Moreover, by the nonnegativity of  $F$  we see that the functionals  $F_r$  are also nonnegative:  $F_r(f, g) \geq 0$  whenever  $f, g \geq 0$ , and this readily implies that the kernels  $\alpha_r(x, y) \geq 0$ . Besides,

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} \alpha_r(x, y) dx dy = F_r(1, 1) \leq C,$$

where  $C = C(X, Y)$  is the constant from (3.1). Therefore, the sequence of nonnegative measures  $\mu_r = \alpha_r(x, y) dx dy$  weakly converges as  $r \rightarrow \infty$  to a finite nonnegative Radon measure  $\mu = \mu(x, y)$ . Since  $X \times Y = \cap_{r=1}^\infty X_r \times Y_r$ , we see that  $\text{supp } \mu \subset X \times Y$ . For  $f \in C(X)$ ,  $g \in C(Y)$  let

$\tilde{f}, \tilde{g} \in C(\mathbb{R}^m)$  be continuous extensions of these functions on the whole space. Then, in view of (3.4), (3.5)

$$\begin{aligned} F(f, g) &= \lim_{r \rightarrow \infty} F_r(\tilde{f}, \tilde{g}) = \lim_{r \rightarrow \infty} \int_{\mathbb{R}^m \times \mathbb{R}^m} \tilde{f}(x) \tilde{g}(y) \alpha_r(x, y) dx dy = \\ &= \int_{X \times Y} \tilde{f}(x) \tilde{g}(y) d\mu(x, y) = \int_{X \times Y} f(x) g(y) d\mu(x, y), \end{aligned}$$

and representation (3.3) follows. Observe that the measure  $\mu$  is finite and uniquely defined by (3.3) because linear combinations of the functions  $f(x)g(y)$  are dense in  $C(X \times Y)$ . Thus, the proof in the case of compacts  $X \subset \mathbb{R}^m$ ,  $Y \subset \mathbb{R}^l$  is complete.

In the case of arbitrary Hausdorff compacts  $X, Y$ , we introduce the set  $\mathfrak{A}$ , consisting of pairs  $(A, B)$  of finite subsets  $A \subset C(X)$ ,  $B \subset C(Y)$ . The set  $\mathfrak{A}$  is ordered by the inclusion order:  $\alpha = (A_1, B_1) \leq \beta = (A_2, B_2)$  if  $A_1 \subset A_2$ ,  $B_1 \subset B_2$ . It is clear, that for each  $\alpha, \beta \in \mathfrak{A}$  there exists  $\gamma \in \mathfrak{A}$  such that  $\alpha \leq \gamma$ ,  $\beta \leq \gamma$ , that is,  $\mathfrak{A}$  is a directed set. Let  $\alpha = (A, B) \in \mathfrak{A}$ ,  $A = \{f_1(x), \dots, f_m(x)\} \subset C(X)$ ,  $B = \{g_1(y), \dots, g_l(y)\} \subset C(Y)$ ,  $m, l \in \mathbb{N}$ , and let  $F : X \mapsto \mathbb{R}^m$ ,  $G : Y \mapsto \mathbb{R}^l$  be continuous mapping such that  $F(x) = (f_1(x), \dots, f_m(x))$ ,  $G(y) = (g_1(y), \dots, g_l(y))$ . Then  $\tilde{X} = F(X)$ ,  $\tilde{Y} = G(Y)$  are compact subsets of Euclidean spaces  $\mathbb{R}^m$  and  $\mathbb{R}^l$ , respectively. We introduce the bilinear functional  $F_\alpha(\phi, \psi)$  on  $C(\tilde{X}) \times C(\tilde{Y})$ , setting

$$F_\alpha(\phi, \psi) = F(\phi(F(x)), \psi(G(y))). \quad (3.6)$$

Clearly, this functional satisfies both the continuity and the nonnegativity conditions. Then, as we have already established, there exists a unique nonnegative Radon measure  $\nu_\alpha = \nu_\alpha(p, q)$  on  $\tilde{X} \times \tilde{Y}$  such that

$$F_\alpha(\phi, \psi) = \int_{\tilde{X} \times \tilde{Y}} \phi(p) \psi(q) d\nu_\alpha(p, q). \quad (3.7)$$

Moreover,  $\nu_\alpha(\tilde{X} \times \tilde{Y}) = F(1, 1) \leq C$ , where  $C = C(\tilde{X}, \tilde{Y})$  is the constant from condition (3.1). We consider the linear functional

$$\varphi_\alpha(h) = \int \tilde{h}(p, q) d\nu_\alpha(p, q), \quad (3.8)$$

defined on the subspace  $H_\alpha$  of  $C(X \times Y)$ , consisting of functions  $h(x, y) = \tilde{h}(F(x), G(y))$ ,  $\tilde{h}(p, q) \in C(\tilde{X} \times \tilde{Y})$ . This functional satisfies the property

$$\varphi_\alpha(h) \leq p(h) \doteq C \max_{X \times Y} h^+(x, y), \quad h^+ = \max(h, 0) \quad (3.9)$$

for all real function  $h \in H_\alpha$ . Observe that  $p(h)$  is a sub-linear functional on  $C(X \times Y)$ . Hence, by Hahn-Banach theorem the functional  $\varphi_\alpha$  can be extended to a linear functional  $\tilde{\varphi}_\alpha$  on the whole space  $C(X \times Y)$ , satisfying estimate (3.9) for real continuous functions on  $X \times Y$ . In particular, for each real  $h(x, y) \in C(X \times Y)$

$$-C \max_{X \times Y} h^-(x, y) = -p(-h) \leq \tilde{\varphi}_\alpha(h) \leq p(h) = C \max_{X \times Y} h^+(x, y),$$

which implies, firstly, that  $\tilde{\varphi}_\alpha(h) \geq 0$  whenever  $h \geq 0$  and, secondly, that  $|\tilde{\varphi}_\alpha(h)| \leq C \max_{X \times Y} |h(x, y)| = C \|h\|_\infty$ . We see that  $\tilde{\varphi}_\alpha$  is a nonnegative continuous functional on



$C(X \times Y)$ , and  $\|\varphi_\alpha\| \leq C$ . By Riesz-Markov representation theorem there exists a unique nonnegative Radon measure  $\mu_\alpha$  on  $X \times Y$  such that

$$\tilde{\varphi}_\alpha(h) = \int_{X \times Y} h(x, y) d\mu_\alpha(x, y), \quad (3.10)$$

and  $\mu_\alpha(X \times Y) \leq C$ . Observe also that in view of (3.6), (3.7), (3.8), and (3.10)

$$F(f, g) = \varphi_\alpha(f(x)g(y)) = \int_{X \times Y} f(x)g(y) d\mu_\alpha(x, y) \quad (3.11)$$

for all  $f \in A$ ,  $g \in B$ . Since the space  $M(X \times Y)$  of bounded Radon measures on  $X \times Y$  (with the total variation as a norm) is dual to  $C(X \times Y)$ , then bounded sets in  $M(X \times Y)$  are weakly precompact. Therefore, there exists an accumulation point  $\mu$  of a net  $\mu_\alpha$ ,  $\alpha \in \mathfrak{A}$  with respect to the weak topology in  $M(X \times Y)$ . Let  $f(x) \in C(X)$ ,  $g(y) \in C(Y)$ , and  $\alpha_0 = (\{f\}, \{g\}) \in \mathfrak{A}$ . Since  $\mu$  is an accumulation point of a net  $\mu_\alpha$ , then there exists a increasing sequence  $\alpha_n = (A_n, B_n) \in \mathfrak{A}$ ,  $n \in \mathbb{N}$ , such that  $\alpha_n > \alpha_0$  and in view of (3.11)

$$F(f, g) = \varphi_{\alpha_n}(f(x)g(y)) = \int_{X \times Y} f(x)g(y) d\mu_{\alpha_n}(x, y) \xrightarrow{n \rightarrow \infty} \int_{X \times Y} f(x)g(y) d\mu(x, y).$$

This relation implies the desired representation (3.3) with the finite nonnegative Radon measure  $\mu$ . Uniqueness of the measure  $\mu$  follows again from the density in  $C(X \times Y)$  of linear combinations of the functions  $f(x)g(y)$ .

Now, we consider the general case of locally compact Hausdorff spaces  $X, Y$ . We introduce the directed set  $\mathfrak{K}$  consisting of pairs  $\alpha = (K, L)$  of compacts  $K \subset X$ ,  $L \subset Y$  and ordered by the inclusion order, i.e.,  $\alpha = (K, L) \leq \alpha_1 = (K_1, L_1)$  if  $K \subset K_1$ ,  $L \subset L_1$ . For each  $\alpha = (K, L) \in \mathfrak{K}$  there exist functions  $a_\alpha(x) \in C_0(X)$ ,  $b_\alpha(y) \in C_0(Y)$  with the following properties  $0 \leq a_\alpha(x) \leq 1$ ,  $0 \leq b_\alpha(y) \leq 1$ , and  $a_\alpha(x) = b_\alpha(y) = 1$  for all  $x \in K$ ,  $y \in L$ . We denote  $X_\alpha = \text{supp } a_\alpha$ ,  $Y_\alpha = \text{supp } b_\alpha$  and define the bilinear functional  $F_\alpha : C(X_\alpha) \times C(Y_\alpha) \rightarrow \mathbb{C}$  by the identity  $F_\alpha(f, g) = F(fa_\alpha, gb_\alpha)$ . It is assumed that the functions  $(fa_\alpha)(x)$ ,  $(gb_\alpha)(y)$  are extended on the whole spaces  $X, Y$ , being zero outside of  $X_\alpha, Y_\alpha$ , respectively. In particular, these functions have compact supports and the functional  $F_\alpha$  is well-defined. Obviously,

$$F_\alpha(f, g) \leq C(X_\alpha, Y_\alpha) \|fa_\alpha\|_\infty \|gb_\alpha\|_\infty \leq C_\alpha \|f\|_\infty \|g\|_\infty, \quad C_\alpha = C(X_\alpha, Y_\alpha)$$

and  $F_\alpha(f, g) = F(fa_\alpha, gb_\alpha) \geq 0$  whenever  $f, g$  are real and nonnegative. Since  $X_\alpha, Y_\alpha$  are compact, then, as it was already established above, there exists a unique nonnegative Radon measure  $\mu_\alpha$  on  $X_\alpha \times Y_\alpha$  such that  $F_\alpha(f, g) = \int_{X_\alpha \times Y_\alpha} f(x)g(y) d\mu_\alpha(x, y)$ . This measure may be considered as a Radon measure on the space  $X \times Y$  with the support in  $X_\alpha \times Y_\alpha$ . Then, for every  $f = f(x) \in C(X_\alpha)$ ,  $g = g(y) \in C(Y_\alpha)$

$$F_\alpha(f, g) = \int_{X \times Y} f(x)g(y) d\mu_\alpha(x, y). \quad (3.12)$$

Let  $K \subset X$ ,  $L \subset Y$  be compact subsets,  $\beta = (K, L) \in \mathfrak{K}$ . Suppose that  $\alpha \in \mathfrak{K}$ . Since the nonnegative function  $a_\beta(x)b_\beta(y) \equiv 1$  on  $K \times L$ , then

$$\begin{aligned} \mu_\alpha(K \times L) &\leq \int_{X \times Y} a_\beta(x)b_\beta(y) d\mu_\alpha(x, y) = F_\alpha(a_\beta, b_\beta) = \\ &F(a_\alpha a_\beta, b_\alpha b_\beta) \leq F(a_\beta, b_\beta) = F_\beta(1, 1) \leq C_\beta, \end{aligned} \quad (3.13)$$

where we use the nonnegativity of  $F$ , that implies the monotonicity of this functional on the sets of nonnegative functions:  $F(f_1, g_1) \geq F(f_2, g_2)$  for all  $f_1, f_2 \in C_0(X)$ ,  $g_1, g_2 \in C_0(Y)$  such that  $0 \leq f_1(x) \leq f_2(x)$ ,  $0 \leq g_1(y) \leq g_2(y)$  (indeed,  $F(f_2, g_2) - F(f_1, g_1) = F(f_2 - f_1, g_2) + F(f_1, g_2 - g_1) \geq 0$ ).

In view of estimates (3.13), the net  $\mu_\alpha$ ,  $\alpha \in \mathfrak{K}$  is bounded in locally convex space  $M_{loc}(X \times Y)$  of locally finite Radon measures (with topology generated by seminorms  $p_\alpha(\mu) = |\mu|(K \times L)$ ,  $\alpha = (K, L) \in \mathfrak{K}$ ,  $|\mu|$  stands for the variation of measure  $\mu$ ).

Since the bounded sets of the space  $M_{loc}(X \times Y)$  (which is dual to  $C_0(X \times Y)$ ) are compact, there exists a weak accumulation point  $\mu \in M_{loc}(X \times Y)$  of the net  $\mu_\alpha$ ,  $\alpha \in \mathfrak{K}$ . Since  $\mu_\alpha \geq 0$  for all  $\alpha \in \mathfrak{K}$ , we claim that  $\mu \geq 0$ . Let  $f(x) \in C_0(X)$ ,  $g(y) \in C_0(Y)$ , and  $\alpha_0 = (\text{supp } f, \text{supp } g) \in \mathfrak{K}$ . Since  $\mu$  is an accumulation point of the net  $\mu_\alpha$ ,  $\alpha \in \mathfrak{K}$ , there exists a increasing sequence  $\alpha_n = (K_n, L_n) \in \mathfrak{K}$ ,  $n \in \mathbb{N}$ , such that  $\alpha_n > \alpha_0$  and

$$F(f, g) = \int_{X \times Y} f(x)g(y) d\mu_{\alpha_n}(x, y) \xrightarrow{n \rightarrow \infty} \int_{X \times Y} f(x)g(y) d\mu(x, y).$$

This implies representation (3.3) and conclude the proof.

The following statement, analogous to the assertions of Propositions 1.1, 1.2, holds.

**Proposition 3.1.** *There exists a family of Radon measures  $\mu = \{\mu^{\alpha\beta}\}_{\alpha, \beta=1}^N$  on  $\Omega \times \mathcal{S}$  such that for all  $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ ,  $\psi(\xi) \in \mathcal{A}$ , and  $\alpha, \beta = 1, \dots, N$*

$$\langle \mu^{\alpha\beta}(x, \eta), \Phi_1(x) \overline{\Phi_2(x)} \hat{\psi}(\eta) \rangle = \text{B} \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(U_r^\alpha \Phi_1)(\xi) \overline{F(U_r^\beta \Phi_2)(\xi)} \psi(\xi) d\xi. \quad (3.14)$$

The matrix-valued measure  $\mu$  is Hermitian and positive semi-definite, i.e., for every  $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N$

$$\mu \zeta \cdot \zeta = \sum_{\alpha, \beta=1}^N \mu^{\alpha\beta} \zeta_\alpha \overline{\zeta_\beta} \geq 0.$$

PROOF. Denote for  $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ ,  $\psi(\xi) \in \mathcal{A}$

$$I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = \text{B} \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi) \overline{F(\Phi_2 U_r^\beta)(\xi)} \psi(\xi) d\xi \quad (3.15)$$

and observe that, by the Buniakovskii inequality and the Plancherel identity,

$$|I^{\alpha\beta}| \leq \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \cdot \lim_{r \rightarrow \infty} [\|U_r^\alpha\|_{L^2(K)} \|U_r^\beta\|_{L^2(K)}], \quad (3.16)$$

where  $K \subset \Omega$  is a compact containing supports of  $\Phi_1$  and  $\Phi_2$ . In view of the weak convergence of sequences  $U_r^\alpha$  in  $L^2(K)$  these sequences are bounded in  $L^2(K)$ . Therefore, for some constant  $C_K$  we have  $\|U_r^\alpha\|_{L^2(K)}^2 \leq C_K$  for all  $r \in \mathbb{N}$ ,  $\alpha = 1, \dots, N$ . Then, it follows from (3.16) that

$$|I^{\alpha\beta}(\Phi_1, \Phi_2, \psi)| \leq C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \quad (3.17)$$

with  $K = \text{supp } \Phi_1 \cup \text{supp } \Phi_2$ . If  $\psi(\xi) \in A_0$ , then by Lemma 2.1 the operator  $B_\psi A_{\Phi_1}$  is compact in  $L^2(\mathbb{R}^n)$ . Hence, the sequences  $B_\psi A_{\Phi_1}(U_r^\alpha) = B_\psi A_{\Phi_1}(U_r^\alpha \chi_K) \rightarrow 0$  in  $L^2(\mathbb{R}^n)$ . Here  $\chi_K(x)$  is the indicator function of the compact  $K = \text{supp } \Phi_1$ . We see that for all  $\alpha = 1, \dots, N$

$$F(\Phi_1 U_r^\alpha)(\xi) \psi(\xi) = F(B_\psi A_{\Phi_1}(U_r^\alpha))(\xi) \xrightarrow{r \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}^n),$$

and, therefore, for all  $\alpha, \beta = 1, \dots, N$

$$\lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi) \overline{F(\Phi_2 U_r^\beta)(\xi)} \psi(\xi) d\xi = 0.$$

In view of (3.15)  $I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = 0$  for  $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$  and all  $\psi(\xi) \in A_0$ . We see that the linear with respect to  $\psi$  functional  $I^{\alpha\beta}(\Phi_1, \Phi_2, \psi)$  is well-defined on factor-algebra  $\mathcal{A} = A/A_0$  and, in view of (3.17), for all  $\psi_0 \in A_0$

$$|I^{\alpha\beta}(\Phi_1, \Phi_2, \psi)| = |I^{\alpha\beta}(\Phi_1, \Phi_2, \psi - \psi_0)| \leq C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi - \psi_0\|_\infty.$$

Therefore,

$$|I^{\alpha\beta}(\Phi_1, \Phi_2, \psi)| \leq C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \inf_{\psi_0 \in A_0} \|\psi - \psi_0\|_\infty = C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_{\mathcal{A}}, \quad (3.18)$$

where

$$\|\psi\|_{\mathcal{A}} = \inf_{\psi_0 \in A_0} \|\psi - \psi_0\|_\infty = \text{ess limsup}_{|\xi| \rightarrow \infty} |\psi(\xi)|$$

is the factor-norm of  $[\psi]$  in  $\mathcal{A}$ . Now, we observe that

$$\int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi) \overline{F(\Phi_2 U_r^\beta)(\xi)} \psi(\xi) d\xi = (A_\psi(\Phi_1 U_r^\alpha), \Phi_2 U_r^\beta)_2, \quad (3.19)$$

where  $(\cdot, \cdot)_2$  is the scalar product in  $L^2 = L^2(\mathbb{R}^n)$ . Let  $\omega(x) \in C_0(\mathbb{R}^n)$  be a function such that  $\omega(x) \equiv 1$  on  $\text{supp } \Phi_1$ . Then

$$A_\psi(\Phi_1 U_r^\alpha) = A_\psi B_{\Phi_1}(\omega U_r^\alpha) = B_{\Phi_1} A_\psi(\omega U_r^\alpha) + [A_\psi, B_{\Phi_1}](\omega U_r^\alpha). \quad (3.20)$$

By the definition of algebra  $\mathcal{A}$ , the operator  $[A_\psi, B_{\Phi_1}]$  is compact on  $L^2$  and since  $\omega U_r^\alpha \rightharpoonup 0$  as  $r \rightarrow \infty$  weakly in  $L^2$ , we claim that  $[A_\psi, B_{\Phi_1}](\omega U_r^\alpha) \rightarrow 0$  as  $r \rightarrow \infty$  strongly in  $L^2$ . Since the sequence  $\Phi_2 U_r^\beta$  is bounded in  $L^2$ , we conclude that  $([A_\psi, B_{\Phi_1}](\omega U_r^\alpha), \Phi_2 U_r^\beta)_2 \rightarrow 0$  as  $r \rightarrow \infty$ . It follows from this limit relation and (3.19), (3.20) that

$$I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = \text{B lim}_{r \rightarrow \infty} (B_{\Phi_1} A_\psi(\omega U_r^\alpha), \Phi_2 U_r^\beta)_2 = \text{B lim}_{r \rightarrow \infty} \int_{\mathbb{R}^n} \Phi_1(x) \overline{\Phi_2(x)} A_\psi(\omega U_r^\alpha)(x) \overline{U_r^\beta(x)} dx.$$

We claim that

$$I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = \tilde{I}^{\alpha\beta}(\Phi_1 \overline{\Phi_2}, \hat{\psi}),$$

where

$$\tilde{I}^{\alpha\beta}(\Phi, \hat{\psi}) = \text{B lim}_{r \rightarrow \infty} \int_{\mathbb{R}^n} \Phi(x) A_\psi(\omega U_r^\alpha)(x) \overline{U_r^\beta(x)} dx$$

is a bilinear functional on  $C_0(\Omega) \times C(\mathcal{S})$  for each  $\alpha, \beta = 1, \dots, N$  ( $\omega(x) \in C_0(\mathbb{R}^n)$  is an arbitrary function equalled 1 on support of  $\Phi(x)$ ), and  $\hat{\psi}(\eta)$  being the Gelfand transform of  $\psi(\xi)$ . Taking in the above relation  $\Phi_1 = \Phi(x)/\sqrt{|\Phi(x)|}$  (we set  $\Phi_1(x) = 0$  if  $\Phi(x) = 0$ ),  $\Phi_2 = \sqrt{|\Phi(x)|}$ , where  $\Phi(x) \in C_0(\Omega)$ , we find with the help of (3.18) that

$$\begin{aligned} |\tilde{I}^{\alpha\beta}(\Phi, \hat{\psi})| &= |I^{\alpha\beta}(\Phi_1, \Phi_2, \psi)| \leq C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_{\mathcal{A}} \\ &= C_K \|\Phi\|_\infty \|\psi\|_{\mathcal{A}} = C_K \|\Phi\|_\infty \|\hat{\psi}\|_\infty, \quad K = \text{supp } \Phi. \end{aligned}$$

This estimate shows that the functionals  $\tilde{I}^{\alpha\beta}(\Phi, \hat{\psi})$  are continuous on  $C_0(\Omega) \times C(\mathcal{S})$ . Now, we observe that for nonnegative  $\Phi(x)$  and  $\hat{\psi}(\eta)$  the matrix  $\tilde{I} \doteq \{\tilde{I}^{\alpha\beta}(\Phi, \hat{\psi})\}_{\alpha, \beta=1}^N$  is Hermitian and positive definite. First, we remark that by Lemma 2.2  $\hat{\psi}(\eta) \geq 0$  if and only if  $\psi(\xi) \geq 0$ . Taking  $\Phi_1(x) = \Phi_2(x) = \sqrt{\Phi(x)}$ , we find

$$\tilde{I}^{\alpha\beta}(\Phi, \hat{\psi}) = I^{\alpha\beta}(\Phi_1, \Phi_1, \psi) = \text{B} \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi) \overline{F(\Phi_1 U_r^\beta)(\xi)} \psi(\xi) d\xi. \quad (3.21)$$

For  $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N$  we have, in view of (3.21),

$$\tilde{I}\zeta \cdot \zeta = \sum_{\alpha, \beta=1}^N \tilde{I}^{\alpha\beta}(\Phi, \hat{\psi}) \zeta_\alpha \overline{\zeta_\beta} = \text{B} \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} |F(\Phi_1 U_r)(\xi)|^2 \psi(\xi) d\xi \geq 0,$$

where  $V_r(x) = \sum_{\alpha=1}^N U_r^\alpha \zeta_\alpha$ . The above relation proves that the matrix  $\tilde{I}$  is Hermitian and positive definite.

We see that for any  $\zeta \in \mathbb{C}^n$  the bilinear functional  $\tilde{I}(\Phi, \hat{\psi})\zeta \cdot \zeta$  is continuous on  $C_0(\Omega) \times C(\mathcal{S})$  and nonnegative, that is,  $\tilde{I}(\Phi, \hat{\psi})\zeta \cdot \zeta \geq 0$  whenever  $\Phi(x) \geq 0$ ,  $\hat{\psi}(\eta) \geq 0$ . By Lemma 3.1 such a functional is represented by integration over some unique locally finite non-negative Radon measure  $\mu = \mu_\zeta(x, \eta) \in M_{loc}(\Omega \times \mathcal{S})$ :

$$\tilde{I}(\Phi, \hat{\psi})\zeta \cdot \zeta = \int_{\Omega \times \mathcal{S}} \Phi(x) \hat{\psi}(\eta) d\mu_\zeta(x, \eta).$$

As a function of the vector  $\zeta$ ,  $\mu_\zeta$  is a measure valued Hermitian form. Therefore,

$$\mu_\zeta = \sum_{\alpha, \beta=1}^N \mu^{\alpha\beta} \zeta_\alpha \overline{\zeta_\beta} \quad (3.22)$$

with measure valued coefficients  $\mu^{\alpha\beta} \in M_{loc}(\Omega \times \mathcal{S})$ , which can be expressed as follows

$$\mu^{\alpha\beta} = [\mu_{e_\alpha + e_\beta} + i\mu_{e_\alpha + ie_\beta}]/2 - (1+i)(\mu_{e_\alpha} + \mu_{e_\beta})/2,$$

where  $e_1, \dots, e_N$  is the standard basis in  $\mathbb{C}^N$ , and  $i^2 = -1$ .

By (3.22)

$$\tilde{I}(\Phi, \hat{\psi})\zeta \cdot \zeta = \sum_{\alpha, \beta=1}^N \langle \mu^{\alpha\beta}, \Phi(x) \hat{\psi}(\eta) \rangle \zeta_\alpha \overline{\zeta_\beta}$$

and since

$$\tilde{I}(\Phi, \hat{\psi})\zeta \cdot \zeta = \sum_{\alpha, \beta=1}^N \tilde{I}^{\alpha\beta}(\Phi, \hat{\psi}) \zeta_\alpha \overline{\zeta_\beta},$$

then, comparing the coefficients, we find that

$$\langle \mu^{\alpha\beta}, \Phi(x) \hat{\psi}(\eta) \rangle = \tilde{I}^{\alpha\beta}(\Phi, \hat{\psi}). \quad (3.23)$$

In particular,

$$\langle \mu^{\alpha\beta}, \Phi_1(x) \overline{\Phi_2(x)} \hat{\psi}(\eta) \rangle = I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = \text{B} \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^\alpha)(\xi) \overline{F(\Phi_2 U_r^\beta)(\xi)} \psi(\xi) d\xi.$$

To complete the proof, observe that for each  $\zeta \in \mathbb{C}^N$  the measure

$$\sum_{\alpha, \beta=1}^N \mu^{\alpha\beta} \zeta_\alpha \overline{\zeta_\beta} = \mu_\zeta \geq 0.$$

Hence,  $\mu$  is Hermitian and positive definite.

The usage of generalized Banach limit instead of extraction of a subsequence is connected with the fact that the algebra  $\mathcal{A}$  is not separable. Therefore, the extraction of a subsequence of  $U_r$  such that relation (3.14) holds, with replacement of the Banach limit to the usual one, is not always possible. Certainly, the  $H$ -measure  $\mu$  depend on the choice of the generalized Banach limit (this resembles the dependence of the Tartar  $H$ -measure on the choice of a subsequence).

If  $X$  is a subspace of  $\mathbb{R}^n$  and  $p_X : \mathcal{S} \rightarrow S_X$  is the projection defined before Proposition 2.1 above, then the image of the measures  $\mu^{\alpha\beta}$  under the map  $(x, \eta) \rightarrow (x, p_X(\eta))$  is exactly the ultraparabolic  $H$ -measure corresponding to the subspace  $X$ .

Evidently, if the sequence  $U_r$  converges as  $r \rightarrow \infty$  to the zero vector strongly in  $L_{loc}^2(\Omega, \mathbb{C}^N)$ , then  $H$ -measure is trivial:  $\mu = 0$ . Conversely, if  $\mu = 0$  then for each  $\Phi(x) \in C_0(\Omega)$

$$\text{B} \lim_{r \rightarrow \infty} \int_{\Omega} |U_r(x) \Phi(x)|^2 dx = \sum_{\alpha=1}^N \text{B} \lim_{r \rightarrow \infty} \int_{\Omega} |F(U_r^\alpha \Phi)(\xi)|^2 d\xi = \sum_{\alpha=1}^N \langle \mu^{\alpha\alpha}(x, \xi), |\Phi(x)|^2 \rangle = 0.$$

This implies that

$$\varliminf_{r \rightarrow \infty} \int_{\Omega} |U_r(x) \Phi(x)|^2 dx = 0 \quad \forall \Phi(x) \in C_0(\Omega). \quad (3.24)$$

We can choose the sequence of real nonnegative functions  $\Phi_k(x) \in C_0(\Omega)$  such that  $\Phi_{k+1}(x) \geq \Phi_k(x)$  for all  $k \in \mathbb{N}$ , and  $\lim_{k \rightarrow \infty} \Phi_k(x) = 1$  for all  $x \in \Omega$ . It follows from (3.24) that there

exists a strictly increasing sequence  $r_k \in \mathbb{N}$  such that  $\int_{\Omega} |U_{r_k}(x) \Phi_k(x)|^2 dx < 1/k$ . Then the subsequence

$$U_{r_k}(x) \xrightarrow[k \rightarrow \infty]{} 0 \quad \text{in } L_{loc}^2(\Omega, \mathbb{C}^N).$$

Let  $\mu = \{\mu^{\alpha\beta}\}_{\alpha, \beta=1}^N$  be an  $H$ -measure corresponding to a sequence  $U_r = \{U_r^\alpha\}_{\alpha=1}^N \in L_{loc}^2(\Omega, \mathbb{C}^N)$ . We define  $\mu_0 = \text{Tr} \mu = \sum_{\alpha=1}^N \mu^{\alpha\alpha}$ . As follows from Proposition 3.1,  $\mu_0$  is a locally finite non-negative Radon measure on  $\Omega \times \mathcal{S}$ . We assume that this measure is extended on  $\sigma$ -algebra of  $\mu_0$ -measurable sets, and in particular that this measure is complete.

**Lemma 3.2.** *The  $H$ -measure  $\mu$  is absolutely continuous with respect to  $\mu_0$ , more precisely,  $\mu = H(x, \eta) \mu_0$ , where  $H(x, \eta) = \{h^{\alpha\beta}(x, \eta)\}_{\alpha, \beta=1}^N$  is a bounded  $\mu_0$ -measurable function taking values in the cone of nonnegative definite Hermitian  $N \times N$  matrices, moreover  $|h^{\alpha\beta}(x, \eta)| \leq 1$ .*

PROOF. Remark firstly that  $\mu^{\alpha\alpha} \leq \mu_0$  for all  $\alpha = 1, \dots, N$ . Now, suppose that  $\alpha, \beta \in \{1, \dots, N\}$ ,  $\alpha \neq \beta$ . By Proposition 3.1 for any compact set  $B \subset \Omega \times \mathcal{S}$  the matrix

$$\begin{pmatrix} \mu^{\alpha\alpha}(B) & \mu^{\alpha\beta}(B) \\ \overline{\mu^{\alpha\beta}(B)} & \mu^{\beta\beta}(B) \end{pmatrix}$$

is nonnegative definite; in particular,

$$|\mu^{\alpha\beta}(B)| \leq (\mu^{\alpha\alpha}(B)\mu^{\beta\beta}(B))^{1/2} \leq \mu_0(B).$$

By regularity of measures  $\mu^{\alpha\beta}$  and  $\mu_0$  this estimate is satisfied for all Borel sets  $B$ . This easily implies the inequality  $\text{Var } \mu^{\alpha\beta} \leq \mu_0$ . In particular, the measures  $\mu^{\alpha\beta}$  are absolutely continuous with respect to  $\mu_0$ , and by the Radon-Nykodim theorem  $\mu^{\alpha\beta} = h^{\alpha\beta}(x, \eta)\mu_0$ , where the densities  $h^{\alpha\beta}(x, \eta)$  are  $\mu_0$ -measurable and, as follows from the inequalities  $\text{Var } \mu^{\alpha\beta} \leq \mu_0$ ,  $|h^{\alpha\beta}(x, \eta)| \leq 1$   $\mu_0$ -a.e. on  $\Omega \times \mathcal{S}$ . We denote by  $H(x, \eta)$  the matrix with components  $h^{\alpha\beta}(x, \eta)$ . Recall that the  $H$ -measure  $\mu$  is nonnegative definite. This means that for all  $\zeta \in \mathbb{C}^N$

$$\mu\zeta \cdot \zeta = (H(x, \eta)\zeta \cdot \zeta)\mu_0 \geq 0. \quad (3.25)$$

Hence  $H(x, \eta)\zeta \cdot \zeta \geq 0$  for  $\mu_0$ -a.e.  $(x, \eta) \in \Omega \times \mathcal{S}$ . Choose a countable dense set  $E \subset \mathbb{C}^N$ . Since  $E$  is countable, then it follows from (3.25) that for a set  $(x, \eta) \in \Omega \times \mathcal{S}$  of full  $\mu_0$ -measure  $H(x, \eta)\zeta \cdot \zeta \geq 0 \ \forall \zeta \in E$ , and since  $E$  is dense we conclude that actually  $H(x, \eta)\zeta \cdot \zeta \geq 0$  for all  $\zeta \in \mathbb{C}^N$ . Thus, the matrix  $H(x, \eta)$  is Hermitian and nonnegative definite for  $\mu_0$ -a.e.  $(x, \eta)$ . After an appropriate correction on a set of null  $\mu_0$ -measure, we can assume that the above property is satisfied for all  $(x, \eta) \in \Omega \times \mathcal{S}$ , and also  $|h^{\alpha\beta}(x, \eta)| \leq 1$  for all  $(x, \eta) \in \Omega \times \mathcal{S}$ ,  $\alpha, \beta = 1, \dots, N$ . The proof is complete.

Now we assume that for all  $\Phi(x) \in C_0^\infty(\Omega)$  the sequence

$$\sum_{\alpha=1}^N \sum_{k=1}^M c_{\alpha k}(x) p_{\alpha k}(\partial/\partial x)(\Phi(x)U_r^\alpha(x)) \xrightarrow{r \rightarrow \infty} 0 \text{ in } L_{loc}^2(\Omega), \quad (3.26)$$

where  $p_{\alpha k}(\partial/\partial x)$  denotes the pseudo-differential operator with symbol  $p_{\alpha k}(\xi) \in A$  and  $c_{\alpha k}(x) \in C(\Omega)$ . Then the  $H$ -measure corresponding to the sequence  $U_r$  satisfy the following localization property.

**Theorem 3.1.** *For each  $\beta = 1, \dots, N$*

$$\sum_{\alpha=1}^N \sum_{k=1}^M c_{\alpha k}(x) \widehat{p_{\alpha k}}(\eta) \mu^{\alpha\beta}(x, \eta) = 0.$$

PROOF. In view of (3.26) for all  $\Phi_1(x), \Phi(x) \in C_0^\infty(\Omega)$  a sequence

$$\sum_{\alpha=1}^N \sum_{k=1}^M \Phi(x) c_{\alpha k}(x) p_{\alpha k}(\partial/\partial x)(\Phi_1(x)U_r^\alpha(x)) \xrightarrow{r \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}^n). \quad (3.27)$$

Since  $p_{\alpha k} \in A$ , then the commutator  $[A_{p_{\alpha k}}, B_{\Phi c_{\alpha k}}]$  is a compact operator in  $L^2(\mathbb{R}^n)$ . Therefore, this operators transform the weakly convergent sequence  $\Phi_1(x)U_r^\alpha(x)$  to the strongly convergent ones, which implies that

$$\sum_{\alpha=1}^N \sum_{k=1}^M [A_{p_{\alpha k}}, B_{\Phi c_{\alpha k}}](\Phi_1(x)U_r^\alpha(x)) \xrightarrow{r \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}^n). \quad (3.28)$$

Putting relations (3.27), (3.28) together, we find

$$\sum_{\alpha=1}^N \sum_{k=1}^M p_{\alpha k}(\partial/\partial x)(\Phi(x)\Phi_1(x)c_{\alpha k}(x)U_r^\alpha(x)) \xrightarrow{r \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}^n).$$

Taking  $\Phi(x)$  in such a way that  $\Phi(x) = 1$  on  $\text{supp } \Phi_1$ , we arrive at the relation

$$\sum_{\alpha=1}^N \sum_{k=1}^M p_{\alpha k}(\partial/\partial x)(\Phi_1(x)c_{\alpha k}(x)U_r^\alpha(x)) \xrightarrow{r \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}^n).$$

Applying the Fourier transformation, we obtain

$$\sum_{\alpha=1}^N \sum_{k=1}^M p_{\alpha k}(\xi)F(\Phi_1 c_{\alpha k} U_r^\alpha)(\xi) \xrightarrow{r \rightarrow \infty} 0 \text{ in } L^2(\mathbb{R}^n). \quad (3.29)$$

We multiply (3.29) by the bounded sequence  $\overline{F(\Phi_2 U_r^\beta)(\xi)\psi(\xi)}$ , where  $\Phi_2(x) \in C_0^\infty(\Omega)$ ,  $\psi(\xi) \in A$ , and  $1 \leq \beta \leq N$ . Integrating over  $\xi \in \mathbb{R}^n$ , we arrive at the relation

$$\lim_{r \rightarrow \infty} \sum_{\alpha=1}^N \sum_{k=1}^M \int_{\mathbb{R}^n} p_{\alpha k}(\xi)F(\Phi_1 c_{\alpha k} U_r^\alpha)(\xi) \overline{F(\Phi_2 U_r^\beta)(\xi)\psi(\xi)} d\xi = 0.$$

On the other hand, by the definition of  $H$ -measure this limit coincides with

$$\begin{aligned} & \text{B} \lim_{r \rightarrow \infty} \sum_{\alpha=1}^N \sum_{k=1}^M \int_{\mathbb{R}^n} p_{\alpha k}(\xi)F(\Phi_1 c_{\alpha k} U_r^\alpha)(\xi) \overline{F(\Phi_2 U_r^\beta)(\xi)\psi(\xi)} d\xi = \\ & \sum_{\alpha=1}^N \sum_{k=1}^M \text{B} \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} p_{\alpha k}(\xi)F(\Phi_1 c_{\alpha k} U_r^\alpha)(\xi) \overline{F(\Phi_2 U_r^\beta)(\xi)\psi(\xi)} d\xi = \\ & \sum_{\alpha=1}^N \sum_{k=1}^M \langle \mu^{\alpha\beta}, c_{\alpha k}(x)\Phi_1(x)\overline{\Phi_2(x)}\widehat{p_{\alpha k}}(\eta)\widehat{\psi}(\eta) \rangle. \end{aligned}$$

Hence,

$$\sum_{\alpha=1}^N \sum_{k=1}^M \langle \mu^{\alpha\beta}, c_{\alpha k}(x)\Phi_1(x)\overline{\Phi_2(x)}\widehat{p_{\alpha k}}(\eta)\widehat{\psi}(\eta) \rangle = 0.$$

This relation can be written in the form

$$\left\langle \sum_{\alpha=1}^N \sum_{k=1}^M c_{\alpha k}(x)\widehat{p_{\alpha k}}(\eta)\mu^{\alpha\beta}(x, \eta), \Phi_1(x)\overline{\Phi_2(x)}\widehat{\psi}(\eta) \right\rangle = 0. \quad (3.30)$$

Since the test functions  $\Phi_1(x), \Phi_2(x) \in C_0^\infty(\Omega)$ ,  $\widehat{\psi}(\eta) \in C(\mathcal{S})$  are arbitrary, the statement of Theorem 3.1 follows from (3.30).

#### 4. Compensated compactness

Assume that a sequence  $u_r(x) \in L_{loc}^2(\Omega, \mathbb{C}^N)$  weakly converges to a vector-function  $u(x)$  as  $r \rightarrow \infty$  while for each  $\Phi(x) \in C_0^\infty(\Omega)$  the sequences

$$\sum_{\alpha=1}^N \sum_{k=1}^M c_{s\alpha k}(x)p_{s\alpha k}(\partial/\partial x)(\Phi(x)u_r^\alpha(x)) \text{ are precompact in } L_{loc}^2(\Omega), \quad (4.1)$$

where  $p_{s\alpha k}(\partial/\partial x)$  are pseudo-differential operators with symbols  $p_{\alpha k}(\xi) \in A$ ,  $c_{s\alpha k}(x) \in C(\Omega)$ , and  $s = 1, \dots, m$ . Introduce the set

$$\Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \eta \in \mathcal{S} : \sum_{\alpha=1}^N \sum_{k=1}^M c_{s\alpha k}(x) \widehat{p_{s\alpha k}}(\eta) \lambda_{\alpha} = 0 \ \forall s = 1, \dots, m \right\}.$$

Now, suppose that

$$q(x, u) = Q(x)u \cdot u = \sum_{\alpha, \beta=1}^N q_{\alpha\beta}(x) u_{\alpha} \overline{u_{\beta}}$$

is an Hermitian form with the matrix  $Q(x)$  of coefficients  $q_{\alpha\beta}(x) \in C(\Omega)$ .

Let the sequence  $q(x, u_r) \rightarrow v$  as  $r \rightarrow \infty$  weakly in  $M_{loc}(\Omega)$ . The following theorem is analogous to Theorem 1.1.

**Theorem 4.1.** *If  $q(x, \lambda) \geq 0$  for all  $\lambda \in \Lambda(x)$ ,  $x \in \Omega$ , then  $q(x, u(x)) \leq v$  ( in the sense of measures ).*

PROOF. Let  $\mu = \{\mu^{\alpha\beta}\}_{\alpha, \beta=1}^N$  be the  $H$ -measure corresponding to the sequence  $U_r = u_r - u$ . By Lemma 3.2 this  $H$ -measure admits the representation  $\mu = H(x, \eta)\mu_0$ , where  $\mu_0 = \text{Tr } \mu \geq 0$  and  $H(x, \eta)$  is a  $\mu_0$ -measurable map from  $\Omega \times \mathcal{S}$  into the cone of nonnegative definite  $N \times N$  Hermitian matrices. As readily follows from (4.1), for each  $\Phi(x) \in C_0^{\infty}(\Omega)$ ,  $s = 1, \dots, m$

$$\sum_{\alpha=1}^N \sum_{k=1}^M c_{s\alpha k}(x) p_{s\alpha k}(\partial/\partial x)(\Phi(x)U_r^{\alpha}(x)) \xrightarrow{r \rightarrow \infty} 0 \text{ strongly in } L_{loc}^2(\Omega).$$

By Theorem 3.1 for all  $s = 1, \dots, m$  and  $\beta = 1, \dots, N$

$$\sum_{\alpha=1}^N \sum_{k=1}^M c_{s\alpha k}(x) \widehat{p_{s\alpha k}}(\eta) H^{\alpha\beta}(x, \eta) \mu_0 = \sum_{\alpha=1}^N \sum_{k=1}^M c_{s\alpha k}(x) \widehat{p_{s\alpha k}}(\eta) \mu^{\alpha\beta}(x, \eta) = 0.$$

This implies that for  $\mu_0$ -a.e.  $(x, \eta)$  the image  $\text{Im } H(x, \eta) \subset \Lambda(x)$ . Since the matrix  $H(x, \eta) \geq 0$ , there exists a unique Hermitian matrix  $R = R(x, \eta) = (H(x, \eta))^{1/2}$  such that  $R \geq 0$  and  $H = R^2$ . By the known properties of Hermitian matrices  $\ker R = \ker H$ , which readily implies that also  $\text{Im } R = \text{Im } H$ . In particular,  $\text{Im } R(x, \eta) \subset \Lambda(x)$  for  $\mu_0$ -a.e.  $(x, \eta) \in \Omega \times \mathcal{S}$ . Let  $\Phi(x) \in C_0(\Omega)$  be a real test function. Then

$$\begin{aligned} \text{B} \lim_{r \rightarrow \infty} \int (\Phi(x))^2 q(x, U_r(x)) &= \sum_{\alpha, \beta=1}^N \text{B} \lim_{r \rightarrow \infty} \int q_{\alpha\beta}(x) \Phi(x) U_r^{\alpha}(x) \overline{\Phi(x) U_r^{\beta}(x)} dx = \\ &= \sum_{\alpha, \beta=1}^N \text{B} \lim_{r \rightarrow \infty} \int F(q_{\alpha\beta} \Phi U_r^{\alpha})(\xi) \overline{F(\Phi U_r^{\beta})(\xi)} d\xi = \sum_{\alpha, \beta=1}^N \langle \mu^{\alpha\beta}, q_{\alpha\beta}(x) (\Phi(x))^2 \rangle = \\ &= \int_{\Omega \times \mathcal{S}} (\Phi(x))^2 \sum_{\alpha, \beta=1}^N q_{\alpha\beta}(x) h^{\alpha\beta}(x, \eta) d\mu_0(x, \eta). \end{aligned} \quad (4.2)$$

Since  $H = R^2$  then

$$h^{\alpha\beta} = \sum_{k=1}^N r_{\alpha k} \overline{r_{\beta k}} \quad \forall \alpha, \beta = 1, \dots, N,$$



where  $r_{ij} = r_{ij}(x, \eta)$ ,  $i, j = 1, \dots, N$  are components of the matrix  $R$ . Therefore,

$$\sum_{\alpha, \beta=1}^N q_{\alpha\beta}(x) h^{\alpha\beta}(x, \eta) = \sum_{k=1}^N \sum_{\alpha, \beta=1}^N q_{\alpha\beta}(x) r_{\alpha k} \overline{r_{\beta k}} = \sum_{k=1}^N q(x, R e_k), \quad (4.3)$$

where  $e_k$ ,  $k = 1, \dots, N$ , is the standard basis in  $\mathbb{C}^N$ . Since  $R(x, \eta)e_k \in \text{Im } R(x, \eta) \subset \Lambda(x)$  for  $\mu_0$ -a.e.  $(x, \eta) \in \Omega \times \mathcal{S}$ , then  $q(x, R(x, \eta)e_k) \geq 0$  for  $\mu_0$ -a.e.  $(x, \eta)$  and it follows from (4.2), (4.3) that

$$\text{B} \lim_{r \rightarrow \infty} \int (\Phi(x))^2 q(x, U_r(x)) \geq 0. \quad (4.4)$$

for all real  $\Phi(x) \in C_0(\Omega)$ . In view of the weak convergence  $u_r \rightharpoonup u$ ,  $q(x, u_r) \rightharpoonup v$  as  $r \rightarrow \infty$ ,

$$q(x, U_r(x)) = q(x, u_r(x)) + q(x, u(x)) - 2 \text{Re}(Q(x) u_r \cdot u) \rightharpoonup v - q(x, u(x)).$$

in  $M_{loc}(\Omega)$ . Now, it follows from (4.4) that

$$\langle v - q(x, u(x)) dx, (\Phi(x))^2 \rangle = \lim_{r \rightarrow \infty} \int (\Phi(x))^2 q(x, U_r(x)) \geq 0$$

and since the real test function  $\Phi(x)$  is arbitrary,  $v \geq q(x, u(x))$ . The proof is complete.

**Corollary 4.1.** *Suppose that  $q(x, \lambda) = 0$  for all  $\lambda \in \Lambda(x)$ . Then  $v = q(x, u(x))$ , that is, the functional  $u \rightarrow q(x, u)$  is weakly continuous.*

PROOF. Applying Theorem 4.1 to the quadratic forms  $\pm q(x, u)$ , we obtain the inequalities  $\pm v \geq \pm q(x, u(x))$ , which readily imply that  $v = q(x, u(x))$ .

#### 4.1. The case of second order differential constraints

Now we assume that the sequences

$$\sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k} (a_{s\alpha k} u_{\alpha r}) + \sum_{\alpha=1}^N \sum_{k,l=1}^n \partial_{x_k x_l} (b_{s\alpha kl} u_{\alpha r}), \quad s = 1, \dots, m, \quad (4.5)$$

are pre-compact in the Sobolev space  $H_{loc}^{-1}(\Omega) \doteq W_{2,loc}^{-1}(\Omega)$ , where the coefficients  $a_{s\alpha k} = a_{s\alpha k}(x)$ ,  $b_{s\alpha kl} = b_{s\alpha kl}(x)$  are supposed to be (generally – complex-valued) continuous functions on  $\Omega$ , and  $b_{s\alpha lk} = b_{s\alpha kl}$ ,  $s = 1, \dots, m$ ,  $\alpha = 1, \dots, N$ ,  $k, l = 1, \dots, n$ .

We denote by  $A_{s\alpha} = A_{s\alpha}(x)$  the vector  $\{a_{s\alpha k}\}_{k=1}^n \in \mathbb{C}^n$  and by  $B_{s\alpha} = B_{s\alpha}(x)$  the symmetric matrices with components  $\{b_{s\alpha kl}\}_{k,l=1}^n$ . Let  $X_s$  be the maximal linear subspace of  $\mathbb{R}^n$  contained in  $\mathbb{R}^n \cap \ker B_{s\alpha}(x)$  for all  $\alpha = 1, \dots, N$  and  $x \in \Omega$ . The following statement easily follows from the definition of the subspace  $X_s$ :

**Lemma 4.1.** *For each  $\alpha = 1, \dots, N$ ,  $\Phi(x) \in C_0(\Omega)$*

$$F(\Phi B_{s\alpha})(\xi) \tilde{\xi} = 0 \quad \forall \xi \in \mathbb{R}^n, \tilde{\xi} \in X_s. \quad (4.6)$$

PROOF. Equality (4.6) readily follows from the relation

$$F(\Phi B_{s\alpha})(\xi) \tilde{\xi} = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \Phi(x) B_{s\alpha}(x) \tilde{\xi} dx = 0$$

because  $B_{s\alpha}(x) \tilde{\xi} = 0$  for all  $x \in \mathbb{R}^n$  by the definition of the subspace  $X_s$ .

We introduce the set

$$\Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \eta \in \mathcal{S} : \sum_{\alpha=1}^N (iA_{s\alpha} \cdot \tilde{\xi}^s(\eta) - B_{s\alpha} \bar{\xi}^s(\eta) \cdot \bar{\xi}^s(\eta)) \lambda_\alpha = 0 \ \forall s = 1, \dots, m \right\}, \quad (4.7)$$

where  $\tilde{\xi}^s(\eta) \in X_s$ ,  $\bar{\xi}^s(\eta) \in X_s^\perp$  are such that  $(\tilde{\xi}^s(\eta), \bar{\xi}^s(\eta)) = p_{X_s}(\eta)$ , and  $p_{X_s} : \mathcal{S} \rightarrow S_{X_s}$  is the projection defined in section 2. Let

$$q(x, u) = Q(x)u \cdot u = \sum_{\alpha, \beta=1}^N q_{\alpha\beta}(x) u_\alpha \bar{u}_\beta$$

be an Hermitian form with coefficients  $q_{\alpha\beta}(x) \in C(\Omega)$ .

Suppose that the sequence  $q(x, u_r) \rightarrow v$  as  $r \rightarrow \infty$  weakly in  $M_{loc}(\Omega)$ . The following theorem is analogous to Theorems 1.1, 4.1.

**Theorem 4.2.** *If  $q(x, \lambda) \geq 0$  for all  $\lambda \in \Lambda(x)$ ,  $x \in \Omega$ , then  $q(x, u(x)) \leq v$ .*

PROOF. We fix  $s \in \overline{1, m}$ , and observe that in view of (4.5) for each  $\Phi(x) \in C_0^\infty(\Omega)$  the distributions

$$\begin{aligned} \sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k} \left( a_{s\alpha k} \Phi u_{\alpha r} - 2 \sum_{l=1}^n b_{s\alpha kl} \Phi_{x_l} u_{\alpha r} \right) + \sum_{\alpha=1}^N \sum_{k,l=1}^n \partial_{x_k x_l} (b_{s\alpha kl} \Phi u_{\alpha r}) = \\ \Phi \left( \sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k} (a_{s\alpha k} u_{\alpha r}) + \sum_{\alpha=1}^N \sum_{k,l=1}^n \partial_{x_k x_l} (b_{s\alpha kl} u_{\alpha r}) \right) + \\ \sum_{\alpha=1}^N \sum_{k=1}^n a_{s\alpha k} \Phi_{x_k} u_{\alpha r} - \sum_{\alpha=1}^N \sum_{k,l=1}^n b_{s\alpha kl} \Phi_{x_k x_l} u_{\alpha r} \end{aligned} \quad (4.8)$$

are pre-compact in the Sobolev space  $H^{-1}(\mathbb{R}^n) \doteq W_2^{-1}(\mathbb{R}^n)$ . Relation (4.8) implies that the sequence

$$\begin{aligned} (1 + |\xi|^2)^{-\frac{1}{2}} \left( \sum_{\alpha=1}^N \sum_{k=1}^n 2\pi i \xi_k \left( F(a_{s\alpha k} \Phi u_{\alpha r})(\xi) - 2 \sum_{l=1}^n F(b_{s\alpha kl} \Phi_{x_l} u_{\alpha r})(\xi) \right) \right. \\ \left. - \sum_{\alpha=1}^N \sum_{k,l=1}^n 4\pi^2 \xi_k \xi_l F(b_{s\alpha kl} \Phi u_{\alpha r})(\xi) \right), \end{aligned} \quad (4.9)$$

$r \in \mathbb{N}$ , is compact in  $L^2(\mathbb{R}^n)$ . Denote  $\tilde{\xi} = P_1 \xi$ ,  $\bar{\xi} = P_2 \xi$ , where  $P_1, P_2$  are the orthogonal projections onto the subspaces  $X_s, X_s^\perp$ , respectively. Multiplying (4.9) by the bounded function  $(1 + |\xi|^2)^{\frac{1}{2}} (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}}$ , we obtain that the sequence

$$\begin{aligned} (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \sum_{\alpha=1}^N (2\pi i (F(A_{s\alpha} \Phi u_{\alpha r})(\xi) - 2F(B_{s\alpha} \nabla \Phi u_{\alpha r})(\xi)) \cdot \xi \\ - 4\pi^2 F(B_{s\alpha} \Phi u_{\alpha r})(\xi) \xi \cdot \xi), \end{aligned} \quad (4.10)$$

$r \in \mathbb{N}$ , is compact in  $L^2(\mathbb{R}^n)$ .

By Lemma 4.1 and symmetricity of the matrix  $F(B_{s\alpha}\Phi u_{\alpha r})(\xi)$  we find that

$$\begin{aligned} F(B_{s\alpha}\Phi u_{\alpha r})(\xi)\xi \cdot \xi &= F(B_{s\alpha}\Phi u_{\alpha r})(\xi)\bar{\xi} \cdot \bar{\xi} + \\ 2F(B_{s\alpha}\Phi u_{\alpha r})(\xi)\tilde{\xi} \cdot \bar{\xi} + F(B_{s\alpha}\Phi u_{\alpha r})(\xi)\tilde{\xi} \cdot \tilde{\xi} &= F(B_{s\alpha}\Phi u_{\alpha r})(\xi)\bar{\xi} \cdot \bar{\xi}, \end{aligned} \quad (4.11)$$

$$F(B_{s\alpha}\nabla\Phi u_{\alpha r})(\xi) \cdot \xi = F(B_{s\alpha}\nabla\Phi u_{\alpha r})(\xi) \cdot (\bar{\xi} + \tilde{\xi}) = F(B_{s\alpha}\nabla\Phi u_{\alpha r})(\xi) \cdot \bar{\xi}. \quad (4.12)$$

Notice also that the sequences

$$\begin{aligned} (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} F(A_{s\alpha}\Phi u_{\alpha r})(\xi) \cdot \bar{\xi}, \\ (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} F(B_{s\alpha}\nabla\Phi u_{\alpha r})(\xi) \cdot \bar{\xi}, \quad r \in \mathbb{N}, \end{aligned} \quad \text{are compact in } L^2(\mathbb{R}^n), \quad (4.13)$$

since the functions  $(1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \bar{\xi}_k$ ,  $k = 1, \dots, n$  lay in the ideal  $A_0$ . It now follows from (4.10)–(4.13) that the sequence of distributions

$$l_r^s = (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \sum_{\alpha=1}^N (2\pi i F(A_{s\alpha}\Phi u_{\alpha r})(\xi) \cdot \tilde{\xi} - 4\pi^2 F(B_{s\alpha}\Phi u_{\alpha r})(\xi) \bar{\xi} \cdot \bar{\xi}),$$

$r \in \mathbb{N}$ , is compact in  $L^2(\mathbb{R}^n)$ . The distributions  $l_r^s$  can be represented as

$$l_r^s = \sum_{\alpha=1}^N \left( \sum_{k=1}^n 2\pi i p_{s\alpha k}(\xi) F(a_{s\alpha k}\Phi u_{\alpha r})(\xi) - \sum_{k,l=1}^n 4\pi^2 q_{s\alpha kl}(\xi) F(b_{s\alpha kl}\Phi u_{\alpha r})(\xi) \right), \quad (4.14)$$

where

$$\begin{aligned} p_{s\alpha k}(\xi) &= (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \tilde{\xi}_k \equiv (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \tilde{\xi}_k \mod A_0, \\ q_{s\alpha kl}(\xi) &= (1 + |\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \bar{\xi}_k \bar{\xi}_l \equiv (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \bar{\xi}_k \bar{\xi}_l \mod A_0. \end{aligned}$$

In particular, we see that  $p_{s\alpha k}(\xi), q_{s\alpha kl}(\xi) \in A_{X_s}$ .

Taking into account compactness of commutators  $[A_\psi, B_\phi]$  in  $L^2(\mathbb{R}^n)$ , where  $(\psi, \phi) = (p_{s\alpha k}(\xi), \chi(x)a_{s\alpha k}(x))$ ,  $(\psi, \phi) = (q_{s\alpha kl}(\xi), \chi(x)b_{s\alpha kl}(x))$ , and  $\chi(x) \in C_0(\Omega)$  is a function such that  $\chi(x)\Phi(x) = \Phi(x)$ , we find that the sequence

$$\chi(x) \sum_{\alpha=1}^N \left( 2\pi i \sum_{k=1}^n a_{s\alpha k}(x) p_{s\alpha k}(\partial/\partial x) - 4\pi^2 \sum_{k,l=1}^n b_{s\alpha kl}(x) q_{s\alpha kl}(\partial/\partial x) \right) (\Phi u_{\alpha r}),$$

$r \in \mathbb{N}$ , is compact in  $L^2(\mathbb{R}^n)$ , that is, the sequence

$$\sum_{\alpha=1}^N \left( 2\pi i \sum_{k=1}^n a_{s\alpha k}(x) p_{s\alpha k}(\partial/\partial x) - 4\pi^2 \sum_{k,l=1}^n b_{s\alpha kl}(x) q_{s\alpha kl}(\partial/\partial x) \right) (\Phi u_{\alpha r}),$$

$r \in \mathbb{N}$ , is compact in  $L_{loc}^2(\Omega)$ . Here  $p_{s\alpha k}(\partial/\partial x)$ ,  $q_{s\alpha kl}(\partial/\partial x)$  are pseudodifferential operators with symbols

$$(|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \tilde{\xi}_k, \quad (|\tilde{\xi}|^2 + |\bar{\xi}|^4)^{-\frac{1}{2}} \bar{\xi}_k \bar{\xi}_l$$

laying in  $A_{X_s} \subset \mathcal{A}$ . Since  $s = 1, \dots, m$ ,  $\Phi(x) \in C_0^\infty(\Omega)$  are arbitrary, we see that our sequence  $u_r$  satisfies constraints of the kind (4.1). Since

$$\widehat{p_{s\alpha k}}(\eta) = \tilde{\xi}_k^s(\eta), \quad \widehat{q_{s\alpha kl}}(\eta) = \bar{\xi}_k^s(\eta) \bar{\xi}_l^s(\eta),$$

the set  $\Lambda = \Lambda(x)$  corresponding to these constraints is

$$\Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \eta \in \mathcal{S} : \sum_{\alpha=1}^N (2\pi i A_{s\alpha} \cdot \tilde{\xi}^s(\eta) - 4\pi^2 B_{s\alpha} \bar{\xi}^s(\eta) \cdot \tilde{\xi}^s(\eta)) \lambda_\alpha = 0 \ \forall s = 1, \dots, m \right\}.$$

Since, in accordance with Remark 2.1,  $\tilde{\xi}^s(t\eta) = a(t, \eta) \tilde{\xi}^s(\eta)$ ,  $\bar{\xi}^s(t\eta) = b(t, \eta) \bar{\xi}^s(\eta)$ ,  $b^2(t, \eta) = ta(t, \eta)$ , then after the transformation  $\eta = (2\pi)^{-1} \eta$  the set  $\Lambda$  will coincide with (4.7). Then the assertion of Theorem 4.2 readily follows from Theorem 4.1. The proof is complete.

#### 4.2. One example

Let us consider the sequence  $u_r = (u_{r1}, u_{r2}, u_{r3}) \in L_{loc}^2(\Omega, \mathbb{C}^3)$ ,  $\Omega \subset \mathbb{R}^3$  weakly convergent to  $u = (u_1, u_2, u_3)$  such that the sequences

$$i(\partial_{x_3} u_{r2} - \partial_{x_2} u_{r3}) + \partial_{x_1}^2 u_{r1}; \quad i(\partial_{x_1} u_{r3} - \partial_{x_3} u_{r1}) + \partial_{x_2}^2 u_{r2}; \quad i(\partial_{x_2} u_{r1} - \partial_{x_1} u_{r2}) + \partial_{x_3}^2 u_{r3}$$

are pre-compact in  $H_{loc}^{-1}(\Omega)$ .

**Theorem 4.3.** *For every pair  $(k, l)$ ,  $1 \leq k < l \leq 3$  we have*

$$u_{rk} \overline{u_{rl}} \xrightarrow{r \rightarrow \infty} u_k \overline{u_l}.$$

PROOF. In the notations of Theorem 4.2 we find that  $X_i = \{\xi \in \mathbb{R}^3 : \xi_i = 0\}$ ,  $i = 1, 2, 3$ , while the set  $\Lambda$  is determined by the relations

$$\begin{aligned} \lambda_2 \tilde{\xi}_3^1(\eta) - \lambda_3 \tilde{\xi}_2^1(\eta) + \lambda_1 (\bar{\xi}_1^1(\eta))^2 &= \lambda_3 \tilde{\xi}_1^2(\eta) - \lambda_1 \tilde{\xi}_3^2(\eta) + \lambda_2 (\bar{\xi}_2^2(\eta))^2 \\ &= \lambda_1 \tilde{\xi}_2^3(\eta) - \lambda_2 \tilde{\xi}_1^3(\eta) + \lambda_3 (\bar{\xi}_3^3(\eta))^2 = 0 \end{aligned} \quad (4.15)$$

for some  $\eta \in \mathcal{S}$ . For  $\gamma \in \mathbb{C}$  we introduce the Hermitian form

$$Q_\gamma(\lambda) = \operatorname{Re} \gamma u_k \overline{u_l} = \frac{\gamma}{2} u_k \overline{u_l} + \frac{\bar{\gamma}}{2} u_l \overline{u_k}.$$

Let  $\lambda \in \Lambda$ . Then there exists  $\eta \in \mathcal{S}$  such that (4.15) holds. Observe that the space  $\tilde{X}$  from Proposition 2.1 may be included at most in two subspaces  $X_i$ . If the set  $I = \{\alpha \in \overline{1, 3} : \tilde{X} \not\subset X_\alpha\}$  contains two different indexes  $j, k$ , then  $\tilde{\xi}^j(\eta) = \tilde{\xi}^k(\eta) = 0$ ,  $|\bar{\xi}^j(\eta)| = |\bar{\xi}^k(\eta)| = 1$  by Proposition 2.1 and it follows from (4.15) that  $\lambda_k = \lambda_j = 0 \Rightarrow Q_\gamma(\lambda) = 0$ .

In the remaining case there exists only one index  $j$  such that  $\tilde{X} \not\subset X_j$ . For definiteness, we assume that  $j = 1$ . Then again  $\tilde{\xi}^1(\eta) = 0$ ,  $\bar{\xi}^1(\eta) \neq 0$ , which imply that  $\lambda_1 = 0$ . It is clear that  $\tilde{X} = X_2 \cap X_3$ . By Proposition 2.1 we find that  $\tilde{\xi}^2(\eta) = (a, 0, 0)$ ,  $\tilde{\xi}^3(\eta) = (b, 0, 0)$ , and  $ab > 0$ . Let  $\bar{\xi}^2(\eta) = (0, p, 0)$ ,  $\bar{\xi}^3(\eta) = (0, 0, q)$ . By (4.15)

$$a\lambda_3 + p^2\lambda_2 = -b\lambda_2 + q^2\lambda_3 = 0.$$

Since the determinant of this system  $\Delta = p^2q^2 + ab > 0$  we conclude  $\lambda_2 = \lambda_3 = 0$ . Thus,  $\lambda = 0$  and  $Q_\gamma(\lambda) = 0$ . By Theorem 4.2 we see that  $Q_\gamma(u_r) \rightarrow Q_\gamma(u)$ . Therefore,

$$u_{rk} \overline{u_{rl}} = Q_1(u_r) - iQ_i(u_r) \xrightarrow{r \rightarrow \infty} Q_1(u) - iQ_i(u) = u_k \overline{u_l},$$

as was to be proved.

Observe that in the notations of Theorem 1.1 the set  $\Lambda = \{\lambda \in \mathbb{R}^3 \mid \lambda_1 \lambda_2 \lambda_3 = 0\}$  and this theorem does not allow to derive the statement of Theorem 4.3.

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